18.1 Coordinate changes and vector integral theorems

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS divergence • conservative field

Objective

Transform double and triple integrals with their Jacobians; Apply Green, Stokes and divergence theorems with correct orientation; Check domain singularities before asserting path independence.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

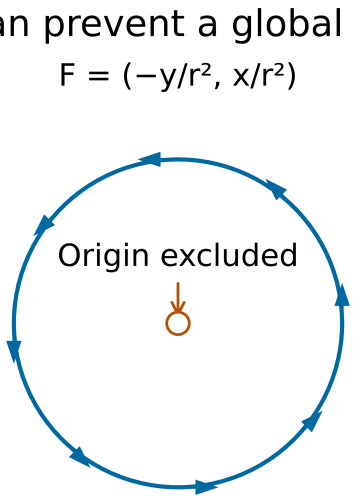
1 Key knowledge and worked examples

divergence : The sum of a vector field's coordinate-wise partial derivatives measuring local outward flow.

conservative field : A vector field equal to a scalar potential gradient with path-independent line integrals.

Changing variables scales area or volume by the absolute Jacobian determinant. Polar coordinates use r dr dtheta; cylindrical coordinates use r dr dtheta dz; spherical coordinates with phi measured from the positive z-axis use $\text{rho}^2 \sin(\text{phi})\, d\rho\, d\text{phi}\, d\text{theta}$. State angle conventions and transform both the integrand and the region. A missing Jacobian changes a uniform density integral into the wrong physical quantity.

For F=(x,y,z), divergence is 3. The outward flux through the sphere of radius 2 is therefore 3 times its volume, or $3 \cdot (4\pi \cdot 2^3/3) = 32\pi$. This avoids a surface parameterisation. For the planar field (−y,x), Green's theorem gives counterclockwise unit-circle circulation as integral of $1-(-1)=2$ over the disk, hence 2π. Reversing the orientation changes the circulation sign.



Zero curl but circulation 2π: the hole at the origin prevents a global potential

2 Practice

2.1 What is the polar-coordinate area element? [1]

- **A** dr dtheta
- **B** r dr dtheta
- **C** r^2 dr dtheta
- **D** sin(theta) dr dtheta
- **E** r^−1 dr dtheta

2.2 For F=(x,y,z), what is the outward flux through the unit sphere? [1]

- **A** 0
- **B** π
- **C** 2π
- **D** 4π
- **E** 12π

2.3 Evaluate $\iint_D (x^2+y^2)\,dA$ on $1 \le x^2+y^2 \le 4$. Find the outward flux of $F=(x,y,z)$ across the unit sphere. [4]

3 Exam-style questions

3.1 Let $G=(-y/(x^2+y^2), x/(x^2+y^2))$ on the punctured plane. Its scalar curl is zero there. Evaluate its circulation on the counterclockwise unit circle. Explain why Green's theorem does not make it zero. [4]

Solutions

2.1 B. The absolute Jacobian of (r cos(theta),r sin(theta)) is r. Unit scale, r squared, sine theta and inverse r do not equal the polar area Jacobian r. The sine factor belongs to spherical coordinates with different variables.

2.2 D. Divergence is 3 and unit-ball volume is $4\pi/3$, so outward flux is 4π. Divergence three times ball volume 4 pi over three is 4 pi. Twelve pi uses sphere area in place of volume; zero and the smaller pi multiples omit part of the flux.

2.3 The annulus has polar bounds $1 \le r \le 2$, $0 \le \theta \le 2\pi$. Including the Jacobian,

$$I = \int_0^{2\pi} \int_1^2 r^3 \, dr \, d\theta = 15\pi/2.$$

F is continuously differentiable on the whole ball and has divergence three. The divergence theorem gives outward flux $3(4\pi/3) = 4\pi$. Inward orientation would reverse its sign.

3.1 Parameterise $r(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$. Then $G(r(t)) = (-\sin t, \cos t) = r'(t)$, so $G \cdot r' = 1$ and the integral is 2π. The field is undefined at the origin inside the disk. Green's theorem requires the needed smoothness on a neighbourhood of the region, not just on its boundary. Zero curl on a punctured domain does not establish a global potential; a closed nonexact field can circulate around its hole.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.