17.1 Multivariable extrema and constrained optimisation

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS Hessian • Lagrange multiplier

Objective

Classify two-variable critical points with the Hessian; Use Lagrange multipliers with a regular constraint; Compare extrema and attainable values under regular constraints.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

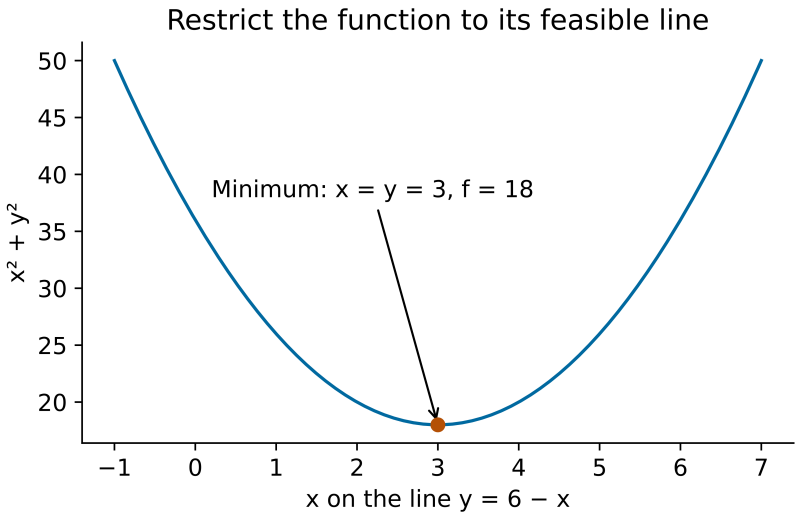
1 Key knowledge and worked examples

Hessian : The matrix of second partial derivatives used to study local curvature.

Lagrange multiplier : A scalar relating objective and regular constraint gradients at a constrained extremum.

For a differentiable function on an open domain, an interior local extremum has zero gradient. This condition is necessary, not sufficient. For two variables at a critical point, let D=f_xx f_yy−(f_xy)^2. If D>0, f_xx>0 gives a strict local minimum and f_xx<0 gives a strict local maximum; D<0 gives a saddle. These tests require suitable second-derivative regularity near the point.

Minimise x^2+y^2 subject to x+y=6. The constraint gradient is (1,1), so 2x=lambda and 2y=lambda. Thus x=y and the constraint gives x=y=3, with value 18. Completing the square on the line gives $x^2+(6-x)^2=2(x-3)^2+18$, proving a global minimum. There is no maximum on this unbounded line, despite the existence of a multiplier candidate.



Substitute the constraint first: the minimum is at $x = y = 3$, where $x^2 + y^2 = 18$

2 Practice

2.1 At a critical point f_xx=2, f_yy=−2 and f_xy=0. What does the second-derivative test give? [1]

- **A** Strict minimum
- **B** Strict maximum
- **C** Saddle point
- **D** Inconclusive
- **E** No critical point

2.2 What is the maximum of xy subject to $x^2+y^2=2$? [1]

- **A** −2
- **B** −1
- **C** 0
- **D** 1
- **E** 2

2.3 Classify the origin for $f(x,y) = x^2 - y^2$ and for $g(x,y) = x^4 + y^4$. State what a zero Hessian determinant permits you to conclude. [4]

3 Exam-style questions

3.1 Find all extrema of $f(x,y) = xy$ subject to $x^2 + y^2 = 2$. Compare with minimising $x^2 + y^2$ on the line $x + y = 4$, including whether maxima exist. [4]

Solutions

2.1 C. D=2(−2)−0^2=−4, so the point is a saddle. Minimum and maximum conflict with the negative Hessian determinant; inconclusive applies when that determinant is zero. The premise explicitly supplies a critical point.

2.2 D. (x−y)^2≥0 implies 2xy≤x^2+y^2=2. Equality occurs at (1,1) and (−1,−1). Negative two and negative one cannot be the maximum because (1,1) gives one. Zero misses positive values; two violates 2xy<=2.

2.3 For f the Hessian is diag$(2, -2)$ with determinant -4; the origin is a saddle. Along the axes the values have opposite signs. For g the Hessian at zero is zero, so the second-derivative test is inconclusive. Nevertheless $x^4 + y^4 \geq 0$, with equality only at the origin; it is a strict global minimum. An inconclusive test is not a classification of the point.

3.1 The circle is compact and its constraint gradient $(2x, 2y)$ never vanishes there. Multipliers give $y = 2\lambda x$, $x = 2\lambda y$. Neither coordinate can be zero at a stationary point; hence $y = \pm x$. The four candidates $(1, 1), (-1, -1), (1, -1), (-1, 1)$ give global maximum 1 and minimum -1. The bound $2|xy| \leq x^2 + y^2 = 2$ checks completeness. On the line, $x^2+(4-x)^2 = 2(x-2)^2+8$, so the minimum is 8 at $(2, 2)$. There is no maximum because the expression is unbounded. Multiplier candidates alone do not establish compactness.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.