

# 17.1 Multivariable extrema and constrained optimisation

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Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 10 marks

KEY WORDS Hessian • Lagrange multiplier

## Objective

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Classify two-variable critical points with the Hessian; Use Lagrange multipliers with a regular constraint; Compare extrema and attainable values under regular constraints.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

## 1 Key knowledge and worked examples

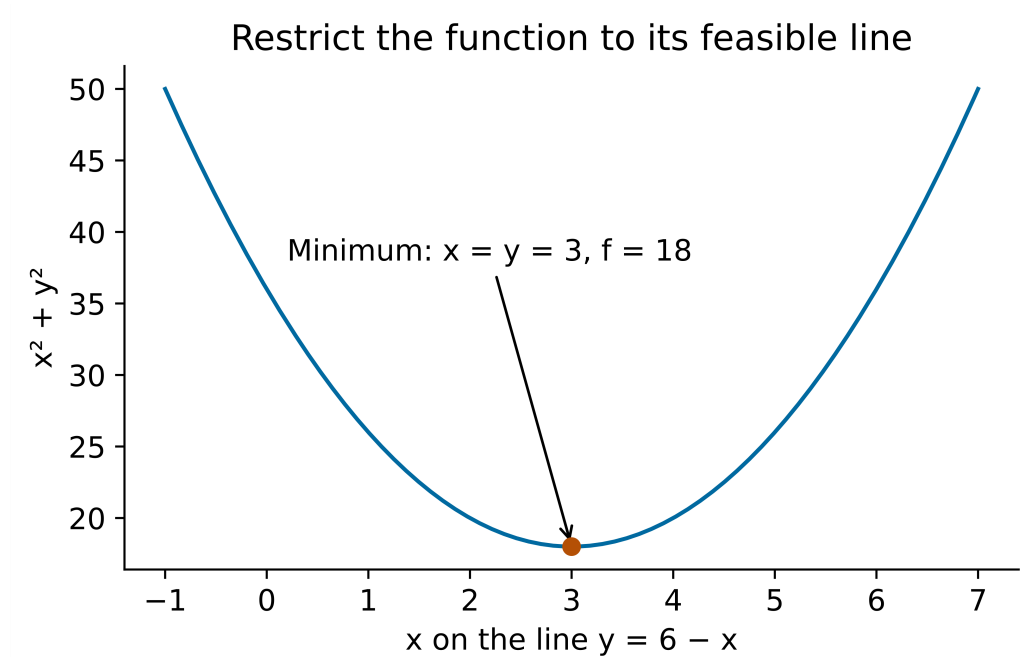
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**Hessian** : The matrix of second partial derivatives used to study local curvature.

**Lagrange multiplier** : A scalar relating objective and regular constraint gradients at a constrained extremum.

For a differentiable function on an open domain, an interior local extremum has zero gradient. This condition is necessary, not sufficient. For two variables at a critical point, let  $D = f_{xx}f_{yy} - (f_{xy})^2$ . If  $D > 0$ ,  $f_{xx} > 0$  gives a strict local minimum and  $f_{xx} < 0$  gives a strict local maximum;  $D < 0$  gives a saddle. These tests require suitable second-derivative regularity near the point.

Minimise  $x^2 + y^2$  subject to  $x + y = 6$ . The constraint gradient is  $(1, 1)$ , so  $2x = \lambda$  and  $2y = \lambda$ . Thus  $x = y$  and the constraint gives  $x = y = 3$ , with value 18. Completing the square on the line gives  $x^2 + (6 - x)^2 = 2(x - 3)^2 + 18$ , proving a global minimum. There is no maximum on this unbounded line, despite the existence of a multiplier candidate.



*Substitute the constraint first: the minimum is at  $x = y = 3$ , where  $x^2 + y^2 = 18$*

## 2 Practice

**2.1** At a critical point  $f_{xx}=2$ ,  $f_{yy}=-2$  and  $f_{xy}=0$ . What does the second-derivative test give? [1]

- **A** Strict minimum
- **B** Strict maximum
- **C** Saddle point
- **D** Inconclusive
- **E** No critical point

**2.2** What is the maximum of  $xy$  subject to  $x^2+y^2=2$ ? [1]

- **A**  $-2$
- **B**  $-1$
- **C**  $0$
- **D**  $1$
- **E**  $2$

**2.3** Classify the origin for  $f(x, y) = x^2 - y^2$  and for  $g(x, y) = x^4 + y^4$ . State what a zero Hessian determinant permits you to conclude. [4]

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### 3 Exam-style questions

**3.1** Find all extrema of  $f(x, y) = xy$  subject to  $x^2 + y^2 = 2$ . Compare with minimising  $x^2 + y^2$  on the line  $x + y = 4$ , including whether maxima exist. [4]

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## Solutions

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**2.1 C.**  $D=2(-2)-0^2=-4$ , so the point is a saddle. Minimum and maximum conflict with the negative Hessian determinant; inconclusive applies when that determinant is zero. The premise explicitly supplies a critical point.

**2.2 D.**  $(x-y)^2 \geq 0$  implies  $2xy \leq x^2 + y^2 = 2$ . Equality occurs at  $(1,1)$  and  $(-1,-1)$ . Negative two and negative one cannot be the maximum because  $(1,1)$  gives one. Zero misses positive values; two violates  $2xy \leq 2$ .

**2.3** For  $f$  the Hessian is  $\text{diag}(2, -2)$  with determinant  $-4$ ; the origin is a saddle. Along the axes the values have opposite signs. For  $g$  the Hessian at zero is zero, so the second-derivative test is inconclusive. Nevertheless  $x^4 + y^4 \geq 0$ , with equality only at the origin; it is a strict global minimum. An inconclusive test is not a classification of the point.

**3.1** The circle is compact and its constraint gradient  $(2x, 2y)$  never vanishes there. Multipliers give  $y = 2\lambda x$ ,  $x = 2\lambda y$ . Neither coordinate can be zero at a stationary point; hence  $y = \pm x$ . The four candidates  $(1, 1)$ ,  $(-1, -1)$ ,  $(1, -1)$ ,  $(-1, 1)$  give global maximum 1 and minimum  $-1$ . The bound  $2|xy| \leq x^2 + y^2 = 2$  checks completeness. On the line,  $x^2 + (4-x)^2 = 2(x-2)^2 + 8$ , so the minimum is 8 at  $(2, 2)$ . There is no maximum because the expression is unbounded. Multiplier candidates alone do not establish compactness.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.