

16.1 Improper integrals and geometric applications

Name: _____ Class: _____ Date: _____

Total: 10 marks**KEY WORDS** improper integral • principal value

Objective

Define improper integrals by limits at each singular boundary; Select area, volume and arc-length formulas from the geometry; Separate convergence of integrals from signed cancellation.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

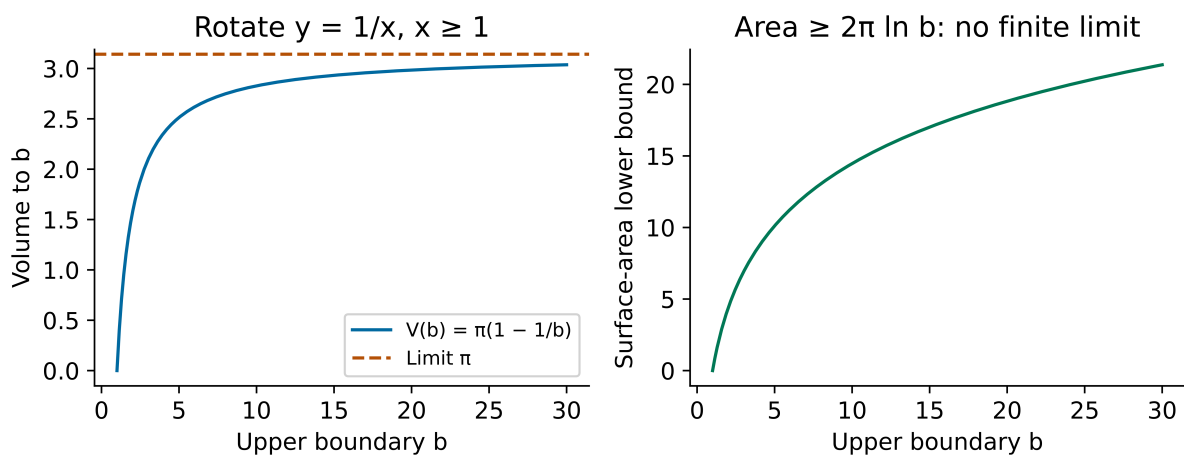
1 Key knowledge and worked examples

improper integral : An integral defined by limits at infinite or singular boundaries.

principal value : A limit using prescribed symmetric cancellation that may exist when an ordinary improper integral diverges.

An infinite integration endpoint is replaced by a finite bound and a limit. At a singular point inside the interval, split the integral and require both one-sided integrals to converge separately. Symmetric cancellation can define a Cauchy principal value but does not prove convergence of the ordinary improper integral. For $1/x$ across zero, the two sides diverge even though symmetric cutoffs cancel.

Rotate $y=1/x$ for $x \geq 1$ around the x -axis. The disk volume is π times the integral of $1/x^2$, so $V=\pi$. The lateral surface integrand is $2\pi(1/x)\sqrt{1+1/x^4}$, at least $2\pi/x$; its improper integral diverges. Finite volume therefore does not imply finite surface area. This comparison avoids trying to find an unnecessary antiderivative.



Gabriel's horn: finite volume π , infinite surface area

2 Practice

2.1 For which p does the integral of x^{-p} from zero to 1 converge? [1]

- **A** $p > 1$
- **B** $p \geq 1$
- **C** $p < 1$
- **D** $p \leq 1$
- **E** All real p

2.2 Rotating the region under $y = x^2$ from $x = 0$ to $x = 1$ around the x -axis gives which volume? [1]

- **A** $\pi/2$
- **B** $\pi/3$
- **C** $\pi/4$
- **D** $\pi/5$
- **E** $2\pi/3$

2.3 For which real p does $\int_0^\infty x^{-p} dx$ converge? Compare the ordinary improper integral of $1/x$ on $[-1, 1]$ with its symmetric principal value. [4]

3 Exam-style questions

3.1 Rotate the region under $y = 1/x$, $x \geq 1$, around the x-axis. Prove its volume is finite and its lateral surface area is infinite. [4]

Solutions

2.1 C. The lower-bound power limit is finite when $1-p>0$; $p=1$ gives logarithmic divergence. Conditions $p>1$ and $p\geq 1$ belong to a different endpoint or include logarithmic divergence. $p\leq 1$ wrongly includes $p=1$; all real p includes divergent examples.

2.2 D. Disk radius is x^2 , so $V=\pi \int_0^1 x^4 dx=\pi/5$. Disk radius is x squared and disk area πx to the fourth. The other fractions result from integrating the wrong power or wrong geometry.

2.3 Split at 1. Near zero, convergence requires $p < 1$; near infinity, it requires $p > 1$. No real p meets both, and $p=1$ diverges logarithmically at both ends. Both one-sided parts must converge independently. For $1/x$ across zero, the negative and positive parts diverge separately, so the ordinary improper integral diverges. Symmetric cutoffs cancel to zero, yielding a principal value only. The zero is not the value of the ordinary integral.

3.1 Disks give $V = \pi \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = \pi \lim(1-1/b) = \pi$. For surface area, $y' = -1/x^2$ and

$$S = 2\pi \int_1^\infty x^{-1} \sqrt{1+x^{-4}} dx.$$

Its nonnegative integrand is at least $2\pi/x$, whose integral diverges. Comparison proves infinite lateral area. This includes neither an end cap nor a claim that finite volume implies finite area.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.