

15.1 Taylor expansions and power-series endpoints

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS radius of convergence • remainder

Objective

Construct Taylor polynomials and control approximation error; Determine radii of convergence and test endpoints separately; Differentiate and integrate power series within their interval of convergence.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

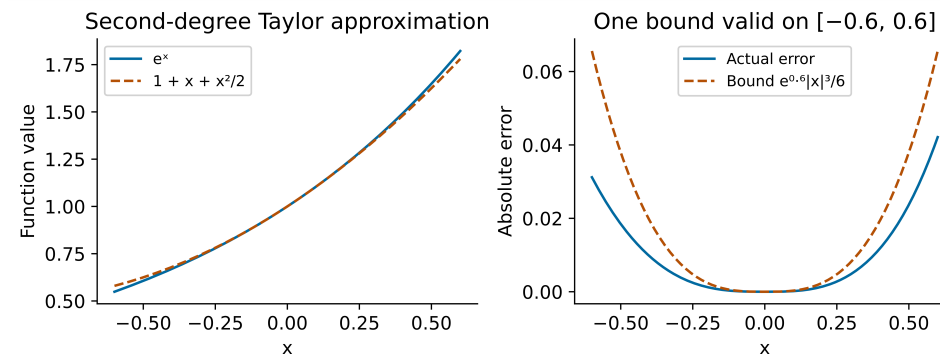
1 Key knowledge and worked examples

radius of convergence : The distance from the series centre inside which a power series converges absolutely.

remainder : The difference between a function and its finite approximation.

The Taylor polynomial of degree m at a is the sum of $f^{(k)}(a)(x-a)^k/k!$ for k from zero to m . The factorial belongs to each coefficient. If the next derivative is bounded in magnitude by M between a and x , the Lagrange remainder has magnitude at most $M|x-a|^{m+1}/(m+1)!$. Smoothness alone does not guarantee that the infinite Taylor series equals the function everywhere.

For $\sum x^n/n$ with $n \geq 1$, the ratio test gives radius 1. At $x=1$ it is the divergent harmonic series; at $x=-1$ it is an alternating convergent series. Thus its real convergence interval is $[-1, 1)$. Differentiating inside gives $\sum x^{n-1} = 1/(1-x)$, which converges at neither endpoint. The radius stayed 1 while the endpoint inclusion changed.



One Lagrange bound covers the whole interval $[-0.6, 0.6]$

2 Practice

2.1 What is the coefficient of x^3 in the Maclaurin series of e^{2x} ? [1]

- A $1/6$
- B $1/3$
- C $2/3$
- D $4/3$
- E 8

2.2 Which is the interval of convergence of $\sum x^n/n^2$ for $n \geq 1$? [1]

- A $(-1, 1)$
- B $[-1, 1)$
- C $(-1, 1]$
- D $[-1, 1]$
- E $\mathbb{R}$

2.3 Find the convergence interval of $\sum_{n=1}^{\infty} x^n/n$ and of its termwise derivative. Approximate $e^{0.1}$ by its quadratic Taylor polynomial with a justified error bound. [4]

3 Exam-style questions

3.1 Evaluate $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$. Explain why replacing cosine by only $1 - x^2/2$ cannot determine this limit. [4]

Solutions

2.1 D. The coefficient is $2^3/3!=8/6=4/3$; the factorial cannot be omitted. One-sixth is the coefficient for e^x ; one-third and two-thirds omit some powers of two; eight omits division by factorial three.

2.2 D. The radius is 1 and both endpoints converge absolutely by comparison with sum $1/n^2$. Each open or half-open interval wrongly discards at least one absolutely convergent endpoint; all of $\mathbb{R}$ includes inputs with magnitude greater than one where terms fail to tend to zero.

2.3 The ratio test gives radius one. At 1 the original harmonic series diverges; at -1 the alternating series converges. Thus the original interval is $[-1, 1)$. The derivative is $\sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$ only for $|x| < 1$; its terms do not tend to zero at either endpoint. $P_2(0.1) = 1+0.1+0.1^2/2 = 1.105$. On $[0, 0.1]$, the third derivative is $e^x < 2$. The Lagrange bound gives $|R_2| \leq 2(0.1)^3/6 = 1/3000$. The bound is deliberately conservative, not a claim of exact error.

3.1 Taylor's theorem near zero gives $\cos x = 1 - x^2/2 + x^4/24 + O(x^6)$. Subtraction cancels the constant and quadratic terms, leaving $x^4/24 + O(x^6)$. After division the limit is $1/24$. A truncation with remainder merely $o(x^2)$ gives no control after division by x^4 ; retain and bound the first surviving order.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.