14.1 Open sets, compactness and connectedness

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS relative topology • connectedness

Objective

Compute closure, interior and boundary in a stated space; Apply compactness and connectedness to continuous maps; Distinguish relative topology from the ambient Euclidean topology.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

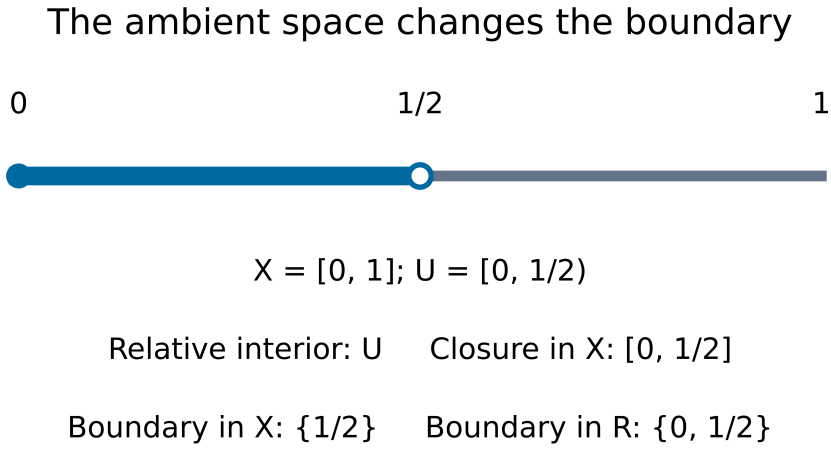
1 Key knowledge and worked examples

relative topology : Open subsets inherited by intersecting a subspace with ambient open sets.

connectedness : Absence of a separation into disjoint nonempty relatively open subsets.

In a metric space, an open set contains a small ball around each of its points. The interior consists of such points; the closure includes all limit points; the boundary is closure minus interior. Open and closed are not mutually exclusive labels; the empty set and the whole space are both. In R, the set [0,1) has interior (0,1), closure [0,1] and boundary {0,1}.

Let X=[0,1] with its usual relative topology. The set U=[0,0.5) equals X intersected with (-1,0.5), so U is open in X. Its closure in X is [0,0.5], and its boundary in X is {0.5}; zero is an interior point relative to X. For a continuous f on X with f(0)<0<f(1), connectedness ensures a zero, while compactness separately ensures attained extrema.



Interior, closure and boundary depend on the space: U's boundary is {1/2} in X but {0, 1/2} in $\mathbb{R}$

2 Practice

2.1 What is the boundary of (0,1) as a subset of R? [1]

- **A** The empty set
- **B** {0}
- **C** {1}
- **D** {0,1}
- **E** [0,1]

2.2 Which subset of R is connected but not compact? [1]

- **A** {0,1}
- **B** [0,1]
- **C** (0,1)
- **D** The empty set
- **E** [0,1] union [2,3]

2.3 In $X = [0, 2]$ with its relative topology, find the interior, closure and boundary of $A = [0, 1)$. Is A open or closed in X? [4]

3 Exam-style questions

3.1 Give a compact disconnected real set and a connected noncompact real set. Must a continuous real function on the latter attain an extreme value? Explain which property gives the intermediate value theorem. [4]

Solutions

2.1 D. The closure is [0,1] and the interior is (0,1); removing the latter leaves both endpoints. Empty omits both endpoints; the singleton options omit one endpoint; [0,1] incorrectly includes interior points.

2.2 C. An open interval is connected but not closed in R, so it is not compact. The other options are compact or disconnected. The finite two-point set and the disjoint closed intervals are disconnected; the closed interval and the empty set are compact. The empty set is connected under the usual convention but still fails noncompactness.

2.3 $A = X \cap (-1, 1)$ is open in X, so its relative interior is A. Its closure is $[0, 1]$ and its boundary is $\{1\}$. Zero is interior relative to X because a sufficiently small ball in X has no negative part. A is not closed in X: the sequence $1 - 1/n$ for $n \geq 2$ tends to the missing point 1. In the ambient real line, interior and boundary would instead be $(0, 1)$ and $\{0, 1\}$.

3.1 The finite set $\{0, 2\}$ is compact but disconnected. The interval $(0, 1)$ is connected but not compact. The continuous identity function on $(0, 1)$ attains neither a maximum nor a minimum. Connectedness makes a continuous real image an interval and therefore supplies intermediate values. Compactness instead makes a nonempty continuous real image attain its extreme values. Neither property substitutes for the other.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.