13.1 Sequences, series and uniform convergence

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS Cauchy sequence • uniform convergence

Objective

Use Cauchy, monotone convergence and subsequence criteria; Distinguish pointwise from uniform convergence of functions; Check hypotheses before interchanging limits and integrals.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

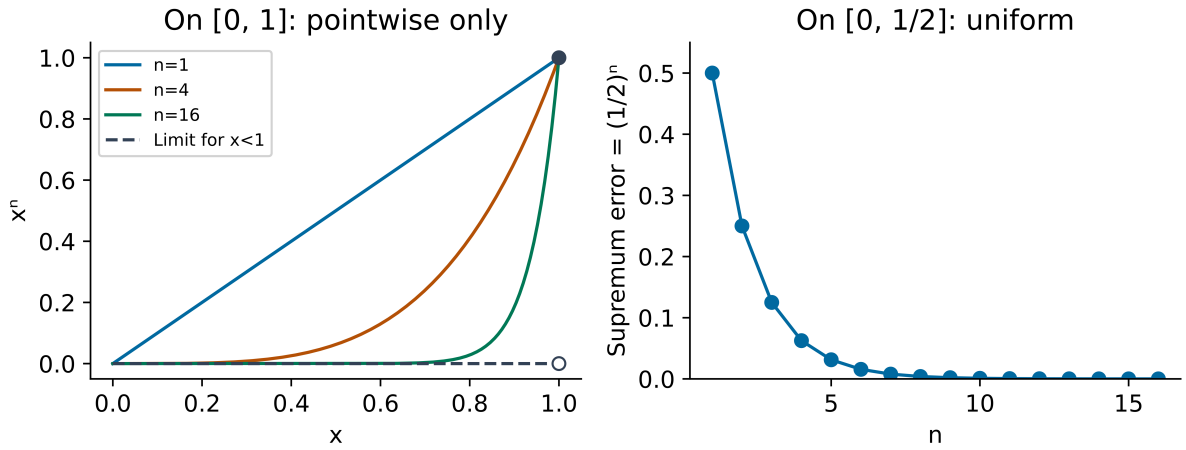
1 Key knowledge and worked examples

Cauchy sequence : A sequence whose sufficiently late terms are arbitrarily close to one another.

uniform convergence : Convergence with one error threshold index valid at every point of the domain.

A convergent real sequence is Cauchy, and every real Cauchy sequence converges because R is complete. In Q a Cauchy sequence can approach an irrational number and fail to converge within Q. A bounded monotone real sequence converges. A bounded sequence need not converge, but Bolzano–Weierstrass guarantees a convergent subsequence; (-1)^n has two different subsequential limits.

On [0,1], f_n(x)=x^n tends to 0 for x<1 and to 1 at x=1. The limit is discontinuous, so convergence cannot be uniform; directly, the supremum error is 1, approached below 1. On [0,a] with 0≤a<1, the limit is zero and the supremum is a^n, which tends to zero, so convergence is uniform. Changing the domain changes the conclusion.



x^n converges only pointwise on [0, 1], but uniformly on [0, 1/2]

2 Practice

2.1 Which statement must hold for a bounded real sequence? [1]

- **A** It converges
- **B** It is monotone
- **C** It has a convergent subsequence
- **D** It is Cauchy
- **E** Its terms tend to zero

2.2 For f_n(x)=sin(nx)/n on R, convergence to zero is what type? [1]

- **A** Not pointwise
- **B** Pointwise but not uniform
- **C** Uniform
- **D** Only at integer x
- **E** Divergent at zero

2.3 Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0,1]$ and on $[0,a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence. [4]

3 Exam-style questions

3.1 Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]

Solutions

2.1 C. Bolzano–Weierstrass gives a convergent subsequence. The sequence (-1)^n rules out the other necessary claims. The bounded sequence (-1)^n is neither convergent, monotone nor Cauchy and does not tend to zero. Its even subsequence does converge.

2.2 C. The supremum of the absolute value is 1/n, independent of x, which tends to zero. A, D and E fail because the uniform error bound includes every real x, including zero and noninteger inputs. B fails because the same bound 1/n tends to zero independently of x.

2.3 On $[0, 1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous f_n cannot converge uniformly. More directly the supremum error on $[0, 1)$ is one. On $[0, a]$ the supremum is $a^n \to 0$, so convergence is uniform to zero. The p-series with p=2 converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

3.1 $\sup_{x\in\mathbb{R}} |g_n(x)| = 1/n \to 0$, so the functions converge uniformly to zero. But $g_n'(x) = \cos(nx)$, and at $x = 0$ these derivatives are all 1. The derivative of the zero limit is 0. Thus uniform convergence of the functions alone does not justify differentiating a limit; additional derivative-convergence hypotheses are needed.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.