

10.1 Rings, ideals and modules

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS ideal • torsion

Objective

Distinguish units, zero divisors and integral domains; Identify ideals and interpret quotient rings; Compare modules over rings with vector spaces over fields; Derive Boolean-ring properties without assuming commutativity.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

ideal : An additive subgroup of a ring absorbing multiplication by all ring elements.

torsion : An element is annihilated by a nonzero scalar in the stated module.

For the domain, ideal and quotient examples below, use a commutative ring with identity 1 distinct from 0. A source problem may specify a general ring instead; do not assume its multiplication commutes unless stated or proved. A unit has a multiplicative inverse. A nonzero zero divisor multiplies some nonzero element to zero. An integral domain has no such zero divisors; cancellation of a nonzero factor then works. A field is a domain in which every nonzero element is a unit. $\mathbb{Z}$ is a domain but not a field; $\mathbb{Z}/6\mathbb{Z}$ is neither.

In $\mathbb{Z}/6\mathbb{Z}$, $2 \cdot 3=0$, so 2 and 3 are zero divisors. The units are 1 and 5 because their gcd with 6 is 1. In $\mathbb{Z}$, the ideal $5\mathbb{Z}$ gives the field $\mathbb{Z}/5\mathbb{Z}$, while $6\mathbb{Z}$ gives a quotient with zero divisors. As $\mathbb{Z}$ -modules, the map from $\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$ has kernel $6\mathbb{Z}$; the quotient identifies integers differing by a multiple of 6, rather than producing a real vector space.

Multiplication in $\mathbb{Z}/6\mathbb{Z}$: units and zero divisors

Residue	Product mod 6	Classification
1, 5	$1 \cdot 1 = 1$; $5 \cdot 5 = 1$	Units
2, 3, 4	$2 \cdot 3 = 0$; $4 \cdot 3 = 0$	Nonzero zero divisors
0	$0 \cdot a = 0$	Zero

In $\mathbb{Z}/6\mathbb{Z}$ a residue is a unit exactly when it is coprime to 6

2 Practice

2.1 How many units are in $\mathbb{Z}/10\mathbb{Z}$? [1]

- A 2
- B 3
- C 4
- D 5
- E 9

2.2 Which integer ideal is maximal in $\mathbb{Z}$? [1]

- A $0\mathbb{Z}$
- B $4\mathbb{Z}$
- C $6\mathbb{Z}$
- D $7\mathbb{Z}$
- E $9\mathbb{Z}$

2.3 In $\mathbb{Z}/12\mathbb{Z}$, list the units and nonzero zero divisors. Is the ideal (4) prime or maximal? [4]

3 Exam-style questions

3.1 As a $\mathbb{Z}$ -module, can $\mathbb{Z}/6\mathbb{Z}$ be free? Compare the ideals (2) and (x) in $\mathbb{Z}$ and $\mathbb{Q}[x]$, respectively, using their quotients. [4]

Solutions

2.1 C. Only 1,3,7,9 are coprime to 10; each has a multiplicative inverse. There are four units. Two or three omit units; five or nine include a residue divisible by two or five.

2.2 D. $\mathbb{Z}/7\mathbb{Z}$ is a field because 7 is prime. The composite quotients have zero divisors, and $\mathbb{Z}/0\mathbb{Z}$ is $\mathbb{Z}$, not a field. The zero ideal has quotient $\mathbb{Z}$, which is not a field. Quotients by four, six and nine have nonzero zero divisors, so those ideals are not maximal.

2.3 Units are the residues coprime to 12: 1, 5, 7, 11. The remaining nonzero residues 2, 3, 4, 6, 8, 9, 10 are zero divisors. For each such a , $a \cdot (12/\gcd(a, 12)) = 0$ modulo 12, with the second factor nonzero. The ideal $(4) = \{0, 4, 8\}$ gives quotient isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This is not a domain, since $2 \cdot 2 = 0$, and is not a field. Hence the ideal is neither prime nor maximal.

3.1 A nonzero free $\mathbb{Z}$ -module has no nonzero vector annihilated by a nonzero integer: examine each integer coordinate. But $6[1] = 0$ in $\mathbb{Z}/6\mathbb{Z}$. It is not free; a field-style basis argument is invalid. $\mathbb{Z}/(2) \cong \mathbb{F}_2$ is a field, so (2) is maximal and prime. Evaluation at zero maps $\mathbb{Q}[x]$ onto $\mathbb{Q}$ with kernel (x) ; its quotient is a field too. Both arguments use proper ideals in commutative rings with identity.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.