Exercise walk-through

# 13.1 Sequences, series and uniform convergence

GRE · GRE Subject Test · GRE Mathematics

# Sequences, series and uniform convergence

- Decide whether a sequence converges, using monotonicity and the **Cauchy sequence** idea.
- Choose and apply a convergence test for a series.
- Separate absolute from conditional convergence.
- Tell pointwise from uniform convergence, and know what uniform convergence preserves.

# Tests for series

- $n$th-term test: if $a_n \not\to 0$, the series diverges. If $a_n \to 0$, nothing follows ($\sum 1/n$ diverges).
- Comparison: compare with $\sum 1/n^p$ (converges for $p > 1$) or a geometric series.
- Ratio test: if $|a_{n+1}/a_n| \to L$, the series converges absolutely for $L < 1$ and diverges for $L > 1$. $L = 1$ decides nothing.
- Integral test for positive decreasing terms; alternating series test for alternating terms that decrease to 0.

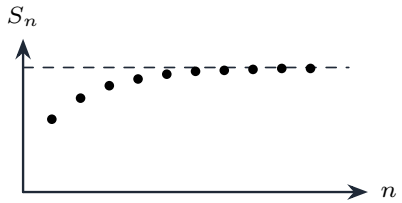
# Worked example: choosing tests

- $\sum \dfrac{n!}{n^n}$: the ratio is $\left(\dfrac{n}{n+1}\right)^n \to \dfrac{1}{e} < 1$, so it converges.
- $\sum \dfrac{\sin nx}{n^2}$: $|\sin nx/n^2| \le 1/n^2$, so by the M-test it converges uniformly on $\mathbb{R}$, and its sum is continuous.
- $\sum \dfrac{(-1)^n}{\sqrt{n}}$: converges by the alternating test, but $\sum 1/\sqrt{n}$ diverges, so the convergence is conditional.

# Exercise 2.1

**2.1** Which statement must hold for a bounded real sequence? [1]

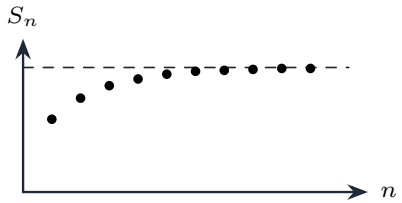
- **A** It converges
- **B** It is monotone
- **C** It has a convergent subsequence
- **D** It is Cauchy
- **E** Its terms tend to zero



# Exercise 2.1

**2.1** Which statement must hold for a bounded real sequence? [1]

- **A** It converges
- **B** It is monotone
- **C** It has a convergent subsequence
- **D** It is Cauchy
- **E** Its terms tend to zero

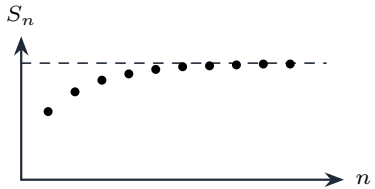



**2.1 C**. Bolzano–Weierstrass gives a convergent subsequence. The sequence (-1)^n rules out the other necessary claims. The bounded sequence (-1)^n is neither convergent, monotone nor Cauchy and does not tend to zero. Its even subsequence does converge.

# Exercise 2.2

**2.2** For f_n(x)=sin(nx)/n on R, convergence to zero is what type? [1]

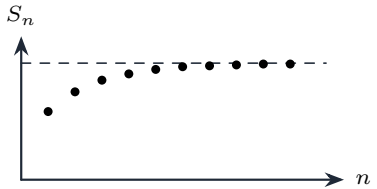
- **A** Not pointwise
- **B** Pointwise but not uniform
- **C** Uniform
- **D** Only at integer x
- **E** Divergent at zero



# Exercise 2.2

**2.2** For f_n(x)=sin(nx)/n on R, convergence to zero is what type? [1]

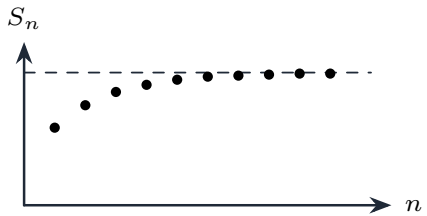
- **A** Not pointwise
- **B** Pointwise but not uniform
- **C** Uniform
- **D** Only at integer x
- **E** Divergent at zero




**2.2 C**. The supremum of the absolute value is 1/n, independent of x, which tends to zero. A, D and E fail because the uniform error bound includes every real x, including zero and noninteger inputs. B fails because the same bound 1/n tends to zero independently of x.

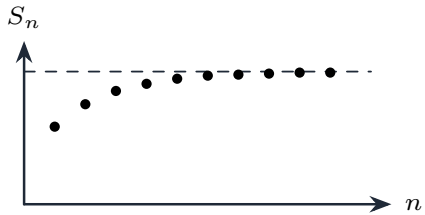
# Exercise 2.3

**2.3** Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0,1]$ and on $[0,a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence. [4]



# Exercise 2.3

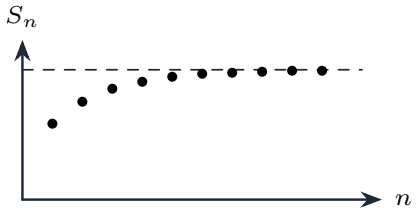
**2.3** Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0,1]$ and on $[0,a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence. [4]




**2.3** On $[0,1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous $f_n$ cannot converge uniformly. More directly the supremum error on $[0,1)$ is one. On $[0,a]$ the supremum is $a^n \to 0$, so convergence is uniform to zero. The p-series with p=2 converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

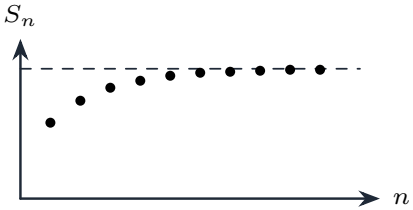
# Exercise 3.1

**3.1** Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]



# Exercise 3.1

**3.1** Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]




**3.1** $\sup_{x\in\mathbb{R}} |g_n(x)| = 1/n \to 0$, so the functions converge uniformly to zero. But $g_n'(x) = \cos(nx)$, and at $x = 0$ these derivatives are all 1. The derivative of the zero limit is 0. Thus uniform convergence of the functions alone does not justify differentiating a limit; additional derivative-convergence hypotheses are needed.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.