Revision practice

GRE Subject Test · GRE Mathematics

Apply the course version boundaries in notes.md; HSK tasks are transferable language practice, not a claimed HSK 3.0 mock.

Question 1

A linear map on R^3 has rank 2. Find its nullity and explain what this means.

Worked answer

Nullity 1 by rank-nullity. Its kernel is a one-dimensional subspace.

Question 2

Evaluate the counterclockwise integral of $4/(z+1)$ on $|z+1|=2$.

Worked answer

$8\pi i$: one enclosed simple pole at -1 has residue 4.

Question 3

Solve $y'=-2y$ with $y(0)=5$ and verify the result by differentiation.

Worked answer

$y=5e^{(-2t)}$; derivative is $-10e^{(-2t)}=-2y$ and initial value is 5.

Question 4

In $Z/24Z$ under addition, find the order of 9 and list the subgroup it generates. How many cosets does it have?

Worked answer

The order is $24/\gcd(9,24)=8$. The subgroup is $\{0,3,6,9,12,15,18,21\}$; it has index $24/8=3$.

Question 5

The additive map $Z/15Z$ to $Z/15Z$ sends x to 5x. Find its kernel and image and explain why the quotient by the kernel is not the entire codomain.

Worked answer

The kernel is $\{0,3,6,9,12\}$, the image is $\{0,5,10\}$, and the quotient has three elements. The codomain has fifteen elements; the map is not onto.

Question 6

In Z/9Z, list the units and nonzero zero divisors. Explain why the ring is not a field despite having an identity.

Worked answer

Units are 1,2,4,5,7,8. Nonzero zero divisors are 3 and 6. Since $3 \cdot 3 = 0$ modulo 9 and 3 has no inverse, not every nonzero element is a unit.

Question 7

Compare 2Z as a submodule of the Z-module Z with 2Z as a proposed subspace of Q. Give a scalar action that distinguishes them.

Worked answer

2Z is closed under integer scaling and addition, so it is a Z-submodule. It is not a Q-vector subspace because rational scalar 1/2 sends 2 to 1, which is not in 2Z.

Question 8

Decide whether x^2+1 is irreducible over Q, R and F2, explaining each change of coefficient field.

Worked answer

Over Q and R it has no root and is an irreducible quadratic. Over F2 it is $(x+1)^2$ and is reducible. The real field differs from the complex field, over which it splits.

Question 9

A finite field extension L/K has degree 10. Can an element of L have minimal polynomial of degree 3 over K?

Worked answer

No. The intermediate field K(alpha) would have degree 3, and the tower law would require 3 to divide 10.

Question 10

Solve $8x \equiv 4$ modulo 12, listing every residue solution in the range 0 through 11.

Worked answer

$\gcd(8,12)=4$, so reduction gives $2x \equiv 1$ modulo 3. Thus $x \equiv 2$ modulo 3 and the four original residues are 2,5,8,11.

Question 11

Find the least nonnegative solution to $x \equiv 3$ modulo 5 and $x \equiv 4$ modulo 7. State the modulus of uniqueness.

Worked answer

Write $x = 3+5k$; $5k \equiv 1$ modulo 7 gives $k \equiv 3$. Therefore $x \equiv 18$ modulo 35; 18 is the least nonnegative solution.

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■ Question 12

Determine whether f_n(x)=x/n converges uniformly to zero on [0,2] and on R.

Worked answer

On [0,2], the supremum error is 2/n, so convergence is uniform. At each fixed real x the limit is still zero, but on R the error is unbounded for every n, so convergence is not uniform there.

■ Question 13

Use the Weierstrass M-test to decide uniform convergence of sum $\sin(nx)/n^2$ on R. Explain why replacing n^2 by n defeats this argument.

Worked answer

Each absolute term is at most $1/n^2$ and that numerical series converges, so the function series converges uniformly and absolutely. The bound 1/n has a divergent sum, making this M-test inconclusive; that failure alone is not a proof of divergence.

■ Question 14

Find the interior, closure and boundary of the rationals Q as a subset of R.

Worked answer

The interior is empty because every interval contains irrational points; the closure is R because rationals are dense. Thus the boundary is all of R.

■ Question 15

In the space X=(0,1), consider the whole set X. Is it relatively closed and relatively open? Is it compact in its usual metric?

Worked answer

The whole space is both open and closed relative to itself. It is not compact; the sequence 1/n for $n \geq 2$ has no subsequence converging to a point in X. Relative closedness alone does not provide Euclidean closedness in R.

■ Question 16

Find the radius and complete real interval of convergence of sum x^n/(n · 2^n), $n \geq 1$.

Worked answer

Radius 2. At -2 the alternating harmonic series converges, while at 2 the harmonic series diverges; the interval is $[-2,2)$.

■ Question 17

Give the degree-three Maclaurin polynomial for sin x and an absolute error bound for $|x| \leq 0.2$ using a degree-three Taylor remainder.

Worked answer

The polynomial is $x - x^3/6$. The fourth derivative is sin x and has absolute value at

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most 1, so the Lagrange bound is $|x|^4/24 \le 0.000066667$. A sharper bound is possible, but is not needed to justify this one.

■ Question 18

Determine whether the integral of $1/x^2$ from -1 to 1 exists as an ordinary improper integral or as a finite symmetric principal value.

Worked answer

Both fail. The integral diverges positively on each side of zero; symmetric cutoffs give 2/epsilon−2, which tends to infinity rather than cancelling.

■ Question 19

Rotate the region between $y=2$ and $y=x$ on $0 \le x \le 2$ around the x-axis. Find its volume using washers.

Worked answer

Outer radius 2 and inner radius x give $V=\pi$ integral 0..2 $(4-x^2)\,dx=\pi(8-8/3)=16\pi/3$.

■ Question 20

Classify the origin for $f=x^2+4xy+y^2$ and $g=x^4+y^4$, explaining which Hessian test is decisive.

Worked answer

For f, f_xx=f_yy=2 and f_xy=4, so $D=4-16=-12$; it is a saddle. For g the Hessian test is inconclusive, but $g \ge 0$ with equality only at the origin proves a strict global minimum.

■ Question 21

Find the maximum and minimum of $x+y$ on $x^2+y^2=8$, stating the points where they occur.

Worked answer

$(x+y)^2 \le 2(x^2+y^2)=16$, so the range is $[-4,4]$. Maximum 4 occurs at (2,2), minimum -4 at $(-2,-2)$. The compact circle ensures extrema are attained.

■ Question 22

Use the divergence theorem to compute the outward flux of $F=(x,2y,3z)$ through the unit cube $0 \le x,y,z \le 1$.

Worked answer

div $F=1+2+3=6$. The closed cube has volume 1, so total outward flux is 6. Opposite face orientations must be respected in a direct calculation.

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Question 23

Evaluate the clockwise unit-circle circulation of $F=(-y,x)$. Explain why Green's theorem cannot be applied over the full disk to $F=(-y/(x^2+y^2),x/(x^2+y^2))$.

Worked answer

For the first field, counterclockwise circulation is 2π; clockwise is -2π. The second field is undefined at the origin and lacks the smoothness hypothesis on the full disk, even though its curl vanishes away from zero.

Question 24

For $f(x)=3x-4$ on R, find its inverse and decide whether f is an involution.

Worked answer

The inverse is $(y+4)/3$. It is not an involution: $f(f(x))=9x-16$ is not identically x.

Question 25

Let $g(x)=2/x$ on R excluding zero. Verify the inverse composition and evaluate $g^9(4)$.

Worked answer

$g(g(x))=2/(2/x)=x$ with every input nonzero. Odd iterates equal g, so $g^9(4)=1/2$.

Question 26

Give a function and disjoint nonempty input sets whose images intersect. State a condition on the function that rules this out.

Worked answer

Use $f(x)=x^2$, $A=\{-3\}$, $B=\{3\}$; both images are $\{9\}$. Injectivity ensures equal outputs require equal inputs, so disjoint input sets then have disjoint images.

Question 27

Negate the claim that every integer has an integer strictly larger than it. Distinguish the negation from a counterexample to the original claim.

Worked answer

The negation is: there exists an integer x such that every integer y satisfies $y\leq x$. The original is true because $y=x+1$ is always an integer; its negation is false, and no counterexample to the original exists.

Question 28

$P(A)=0.6$, $P(B)=0.4$ and $P(A\cap B)=0.24$. Find both conditional probabilities and test independence.

Worked answer

$P(A|B)=0.6$; $P(B|A)=0.4$. The joint probability is the product $0.6\cdot 0.4$, so the events are independent.

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■ Question 29

A sample mean has standard error 4 at n=9 for iid observations. How large must n be to reduce standard error to 1?

Worked answer

Population standard deviation is $4\sqrt{9}=12$. Solve $12/\sqrt{n}=1$ to obtain $n=144$; a fourfold precision gain needs sixteen times as many observations.

■ Question 30

A similar model uses 80% of every original length. Find the area and volume decreases as percentages.

Worked answer

Length factor 0.8 gives area factor 0.64 and volume factor 0.512. The decreases are 36% and 48.8%, respectively.

■ Question 31

Foci are $(-5,0),(5,0)$. Classify the locus r_left+r_right=14 and the locus r_left−r_right=−6; specify any branch.

Worked answer

The first is an ellipse with $a=7, c=5, b^2=49-25=24$: $x^2/49+y^2/24=1$. The second is a hyperbola $x^2/9-y^2/16=1$ restricted to $x\le -3$, because the signed difference is negative.

■ Question 32

For $y=3\cos(4x-\pi)$, give the amplitude, period and horizontal shift. Find one maximum location.

Worked answer

Amplitude 3; period $\pi/2$; right shift $\pi/4$. One maximum is $x=\pi/4$, where the argument is zero.

■ Question 33

For $x=t, y=t^2$ with t in $[-1,1]$, give the traced set, direction and slope at $t=-1/2$.

Worked answer

Only the parabola segment $y=x^2$, $-1\le x\le 1$, is traced. It moves from $(-1,1)$ through $(0,0)$ to $(1,1)$. The slope is $2t$, hence -1 at $t=-1/2$.

■ Question 34

Find the projection of (2,1) onto the line spanned by (1,1), and verify its remainder is orthogonal.

Worked answer

Coefficient is $(2+1)/(1+1)=3/2$, so projection is $(3/2,3/2)$. Remainder $(1/2,-1/2)$ has dot product zero with (1,1).

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■ Question 35

Compute signed and unsigned volumes for edges (1,0,0),(0,2,0),(0,0,3). What changes if the last two edges are exchanged?

Worked answer

The scalar triple product is 6 and volume is 6. Exchanging the last two edges changes the signed result to −6, while geometric volume remains 6.

■ Question 36

Solve y′+2y=6 with y(0)=1 and check the equation.

Worked answer

Integrating factor e^(2x) gives y=3+C e^(−2x); C=−2, so y=3−2e^(−2x). Its derivative is 4e^(−2x), and y′+2y=6.

■ Question 37

Give the general solution of y″−2y′+y=e^x, including the resonant particular term.

Worked answer

The repeated characteristic root is 1, giving $(C_1+C_2x)e^x$. A particular solution is $x^2e^x/2$, because $(D-1)^2(e^x z)=e^x z''$ and z″=1. Add both contributions.

■ Question 38

Count allocations of 10 identical tokens to 3 named boxes when the first gets at least 2 and the others may be empty.

Worked answer

Subtract the first-box lower bound 2. There are 8 remaining stars and 2 bars, so C(10,2)=45 allocations.

■ Question 39

Explain why $S=m^2$ is preserved when S increases by B and then B increases by 2, starting S=0,B=1. Give the state after four iterations.

Worked answer

Before an iteration B=2m+1. Updating S gives $m^2+2m+1=(m+1)^2$, then B becomes 2(m+1)+1. After four iterations S=16,B=9; B is the next addend, not the last one.

■ Question 40

For x=u+2v,y=3u+v, compute v_x and u_y.

Worked answer

J=[[1,2],[3,1]] has determinant −5; inverse is [[−1/5,2/5],[3/5,−1/5]]. Therefore v_x=3/5 and u_y=2/5.

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■ Question 41

Why does the map $(u,v)\mapsto(u^3,v)$ contradict the claim that zero Jacobian determinant means no inverse exists?

Worked answer

It is bijective on $\mathbb{R}^2$ with inverse $(x,y)\mapsto$(real cube root of x,y). Its determinant $3u^2$ vanishes at $u=0$, where the inverse fails to be differentiable. Nonzero determinant is a sufficient differentiable-local-inverse condition, not a necessary condition for mere invertibility.

■ Question 42

For $L=[[1,0],[2,1]]$, $U=[[3,1],[0,2]]$, $b=(7,20)$, solve $LUx=b$.

Worked answer

Forward $y_1=7, y_2=20-14=6$. Backward $x_2=3, x_1=(7-3)/3=4/3$. Verify $Ux=(7,6)$ and $Ly=(7,20)$.

■ Question 43

Maps $A,B:\mathbb{R}^5\to\mathbb{R}^5$ have nullities 3 and 1, respectively. Find bounds on nullity AB and explain why it cannot be 1.

Worked answer

Rank $B=4$. The intersection of im B (dimension 4) with ker A (dimension 3) has dimension at least $4+3-5=2$ and at most 3. Therefore nullity AB ranges from $1+2=3$ to $1+3=4$; the ambient dimension forces overlap.

■ Question 44

Reverse and evaluate $\int_0^1\int_x^1 e\hat{}(2y)dy\,dx$.

Worked answer

Region $0\le x\le y\le 1$ gives $\int_0^1 y\ e\hat{}(2y)dy$. Integration by parts gives $[y\ e\hat{}(2y)/2-e\hat{}(2y)/4]_0^1=(e^2+1)/4$. The factor y comes from the inner x-interval length.

■ Question 45

Evaluate $\int_0\hat{}\pi \sin(8x)/\sin x\,dx$ and justify treatment of both endpoints.

Worked answer

Reflection $x\mapsto\pi-x$ reverses the integrand sign. The endpoint limits are 8 and -8, so singularities are removable and the integral is 0.

■ Question 46

A sphere has $R=2$, depth $h=1$ increasing at $1/2$ per time unit. Find the signed volume rate.

Worked answer

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Current section area is $\pi(4-1)=3\pi$, so signed $dV/dt=3\pi/2$, positive for filling.

■ Question 47

For $f(x)=\sin(4x)$, $g(x)=5x$, give limits of f/g and $(f^2-f)/(2g-g^3)$ as $x\to 0$.

Worked answer

The first is $f'(0)/g'(0)=4/5$. The second is $-f'(0)/(2g'(0))=-2/5$ after factoring, with all nearby denominators checked.

■ Question 48

In the divisor basis on integers at least 2, compute closure of $\{12\}$ and test membership of 6 and 36.

Worked answer

Closure is $\{12,24,36,\ldots\}$. The point 36 belongs; 6 does not, because U_6 excludes 12.

■ Question 49

For $d(x,y)=|x^3-y^3|$ on R, is x_n equal to the cube root of n a Cauchy sequence? Explain using the image.

Worked answer

No. Its image is n, not a Cauchy sequence in ordinary R. The space is complete, but completeness does not make every sequence Cauchy.

■ Question 50

Find the residue of $e\hat{}(2z)/(z-1)^3$ at $z=1$ and its positively oriented integral around a circle centred at 1 containing no other singularity.

Worked answer

For a triple pole, residue is $g''(1)/2!=4e^2/2=2e^2$. The contour integral is $2\pi i$ times the residue, or $4\pi i\, e^2$.

■ Question 51

Count conjugacy classes in S5 by listing the partitions of 5.

Worked answer

The partitions are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1 and 1+1+1+1+1. There are seven classes. Orders alone do not distinguish every type.

■ Question 52

In a general ring where every element is idempotent, prove $2a=0$ without assuming that distinct elements commute.

Worked answer

Apply idempotence to $a+a$: $(a+a)^2=a+a$. Expanding gives $4a^2=2a$; since $a^2=a$, subtract $2a$ to obtain $2a=0$. Only distributivity and additive-ring operations were used.

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■ Question 53

Primitive eighth roots are the roots of x^4+1. Find their sum and product, and justify why these roots have exact order eight.

Worked answer

Their fourth power is -1, so their eighth power is 1 and their order cannot divide 4. The divisors of 8 are 1,2,4,8, so their order is 8. Vieta gives sum 0 and product 1.

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