

GAC010 Mathematics II: Probability, Statistics and Finance

GAC010-aligned practice with invented teaching conditions; the centre's brief sets the actual assessment.

Rules, calculations, and evidence limits

A cheaper headline rate need not produce the cheaper total, and a larger survey need not produce a more representative result.

The method, every time

- Show the known data, why the formula applies, the symbolic equation and the substitution.
- State the unit and interpret the result.
- A statistical summary describes a defined set of observations or a stated model.
- It does not automatically establish a cause or a whole-population finding.

Invented teaching conditions

- All financial amounts, rates, fees and pay rules in the examples are invented teaching conditions.
- Use their stated timing and rounding assumptions.
- Do not assume an actual tax or lending rule.

Two sequences, two growth models

Add the same difference

Rows: 4, 7, 10, 13, 16. Each new row adds three dots.

$$u_n = a + (n - 1)d$$

$$u_5 = 4 + (5 - 1)(3) = 16 \text{ dots.}$$

Multiply by the same ratio

Sheet 2.1 sequence: 3, 6, 12, 24, 48, 96.
Each term doubles.

$$u_n = ar^{n-1}$$

$$u_6 = 3(2)^5 = 96.$$

The sums, and the sum to infinity

Series	Sum, and its condition
Arithmetic	$S_n = \frac{n}{2}[2a + (n - 1)d]$
Geometric, $r \neq 1$	$S_n = \frac{a(1 - r^n)}{1 - r}$; if $r = 1$, $S_n = na$.
Infinite geometric	$S_\infty = \frac{a}{1 - r}$ for $ r < 1$ (nonzero a).

Worked example: the fifth row

row 1: 4 • • • •

row 2: 7 • • • • • • •

row 3: 10 • • • • • • • • • •

- Known: row counts 4, 7 and 10 continue with the same difference.
- Thus $a = 4$ and $d = 3$.
- For the fifth row, use the arithmetic term formula.

$$u_n = a + (n - 1)d; \quad u_5 = 4 + (5 - 1)(3) = 16$$

Worked example: the first five rows together

- 1 Known and required** Row counts have $a = 4$, $d = 3$; find the total for $n = 5$ rows.
- 2 Why a sum** Every row contributes; the fifth term alone is not the total.
- 3 Symbols** $S_n = \frac{n}{2}[2a + (n - 1)d]$
- 4 Substitution** $S_5 = \frac{5}{2}[2(4) + (5 - 1)(3)] = 50$ dots.

Decision

One row has 16 dots; the first five rows together have 50.

Your turn: term or total

Your turn

With $a = 4$ and $d = 3$, which formula gives the total number of dots in the first 6 rows?

- A. The term formula $u_6 = a + 5d$
- B. The sum formula $S_6 = \frac{6}{2}[2a + 5d]$
- C. The geometric sum with $r = 3$
- D. The sum to infinity

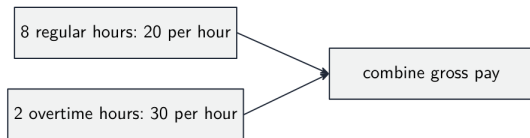
Check: term or total

- 1 Read the task** First six rows together asks for a sum.
- 2 Symbols** $S_n = \frac{n}{2}[2a + (n - 1)d]$
- 3 Substitute** $S_6 = \frac{6}{2}[2(4) + (6 - 1)(3)] = 69$ dots.
- 4 Contrast** $u_6 = a + (6 - 1)d = 4 + 5(3) = 19$ dots describes row six alone.

The vocabulary of pay

Term	Meaning
Gross pay	What you earn before deductions
Net pay	What arrives after deductions such as tax and insurance
Simple interest	$I = Prt$ — interest on the original amount only
Compound interest	$A = P(1 + r)^n$ — interest on the interest
Depreciation	A reduction in recorded value; a fixed-percentage model uses $V_n = V_0(1 - r)^n$

Worked example: regular and overtime hours



- Known: 8 hours at 20 units per hour, and 2 overtime hours at 1.5 times that rate.
- Separate the two categories before adding pay.
- Then take the invented fixed deduction of 15 units once.

$$G = h_r p + h_o k p = 8(20) + 2(1.5)(20) = 220 \text{ units}$$

$$N = G - D = 220 - 15 = 205 \text{ units}$$

Successive discounts are multiplications

- 1 **Track the changing base** The handout uses successive discounts of 20% and 10%.
- 2 **Symbols** $P_{final} = P_0(1 - d_1)(1 - d_2)$
- 3 **Substitute** $P_{final} = P_0(1 - 0.20)(1 - 0.10) = 0.72P_0$
- 4 **Interpret** $d_{total} = 1 - P_{final}/P_0 = 1 - 0.72 = 0.28 = 28\%$.

Decision

The second discount removes 10% of the reduced price, not of the original price.

Your turn: the pay slip

Your turn

6 hours at 15 units plus 4 overtime hours at 1.5x, then a 10-unit deduction. What is net pay?

- A. 90 units
- B. 170 units
- C. 180 units
- D. 160 units

Check: the pay slip

- 1 **Gross pay** $G = h_r p + h_o k p = 6(15) + 4(1.5)(15) = 180$ units.
- 2 **Net pay** $N = G - D = 180 - 10 = 170$ units.
- 3 **Check the distractors** 90 stops after regular pay; 180 omits the deduction; 160 deducts 10 twice.

Present value and the effective rate

A future amount today

For a constant rate and stated timing,
 $FV = PV(1 + r)^n$.

Rearrange: $PV = FV/(1 + r)^n$.

Match r to the period counted by n .

Cash flows change the model

An annuity specifies regular payments. A loan may have an adjusted final payment.

Interest-before-payment:

$$B_n = B_{n-1}(1 + r) - M.$$

APR conventions and included fees must be stated; a label alone is not a schedule.

A nominal rate is not an effective rate

- 1 **Known** Nominal annual rate $j = 0.12$, with $m = 12$ monthly compounding periods; no fees or other cash flows.
- 2 **Monthly rate** $r_m = j/m = 0.12/12 = 0.01$
- 3 **Symbols** $r_{eff} = (1 + r_m)^m - 1$
- 4 **Substitute** $r_{eff} = (1.01)^{12} - 1 \approx 0.126825 = 12.68\%$.

Decision

Twelve successive 1% increases compound; they do not add to an effective 12%.

Worked example: interest then payment



- Known: opening balance 1000 units, period rate 2%, end-of-period payment 300 units.
- Under the stated convention, interest comes before payment.
- Each new period's interest uses the new balance, not the original principal.

$$B_1 = B_0(1 + r) - M = 1000(1.02) - 300 = 720$$

$$B_2 = B_1(1 + r) - M = 720(1.02) - 300 = 434.40$$

Your turn: the second period

Your turn

Balance 720 after period 1. Interest 2%, payment 300. What closes period 2?

- A. 434.40 units
- B. 420.00 units
- C. 440.00 units
- D. 720.00 units

Check: the second period

- 1 Use the running balance** Period two opens at 720 units; interest is added before payment.
- 2 Symbols** $B_2 = B_1(1 + r) - M$
- 3 Substitute** $B_2 = 720(1.02) - 300 = 434.40$ units.
- 4 Wrong-base check** Using interest on the original 1000 would give $720 + 1000(0.02) - 300 = 440$ units, not 434.40.

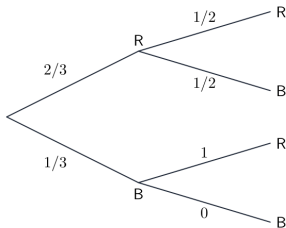
The probability rulebook

Rule	Statement
Equally likely	$P(A) = \text{favourable}/\text{total}$
Mutually exclusive	$P(A \cup B) = P(A) + P(B)$
Independent	$P(A \cap B) = P(A)P(B)$
Complement	$P(A^c) = 1 - P(A)$
Conditional	$P(A \mid B) = P(A \cap B)/P(B),$ $P(B) > 0$

Use the evidence

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ when overlap is possible.

Worked example: two draws without replacement



- Known: two red and one blue counter; two draws without replacement.
- For two reds, follow the red-red path and multiply.

$$P(RR) = P(R_1)P(R_2 | R_1) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

Your turn: at least one red

Your turn

Same bag, two draws without replacement. What is the probability of at least one red?

- A. $1/3$
- B. 1
- C. $2/3$
- D. $1/6$

Check: at least one red

- 1 **Complement event** No red in two draws means two blues.
- 2 **Count what remains** There is only one blue; without replacement, two blues are impossible.
- 3 **Symbols and substitution** $P(\text{at least one red}) = 1 - P(BB) = 1 - 0 = 1.$

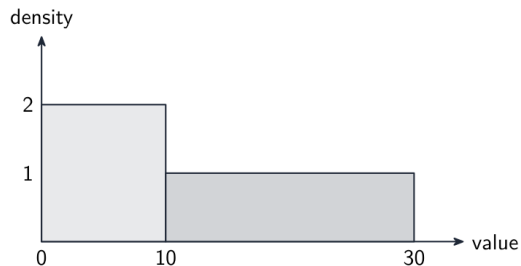
Population, sample, bias

Collection choice	What the conclusion can support
Target: whole school	Define whose characteristic is being estimated.
Only one class surveyed	Describe that class; school-wide claims require more justification.
Only volunteers respond	Selection and nonresponse can distort the result.
Leading question	Wording can change answers; a larger count does not repair it.

Use the evidence

Describe selection and nonresponse before generalising.

Worked example: equal areas, unequal heights



$$D = \frac{f}{w}; \quad D_1 = \frac{20}{10} = 2; \quad D_2 = \frac{20}{20} = 1$$

- Known: class 0–10 has frequency 20; class 10–30 has frequency 20.
- Widths differ, so calculate frequency density: $D = f/w$.
- The second bar is half as high but twice as wide; both areas represent 20 observations.

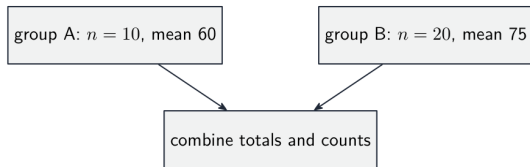
Mean, median, mode

Measure	What it is, and what it costs
The mean	Total over count — uses every value, and every outlier
The median	The middle value in order — an extreme may move without changing it
The mode	The most frequent value — works for categories too; may be absent or repeated

Use the evidence

A high extreme can pull the mean above the median; neither says most observations equal it.

Worked example: a combined mean needs totals



- Known: group A has 10 scores with mean 60; group B has 20 with mean 75.
- Recover each group's total before combining.
- The unweighted mean of 60 and 75 would be 67.5, ignoring the unequal counts.

$$\bar{x} = \frac{n_A \bar{x}_A + n_B \bar{x}_B}{n_A + n_B} = \frac{10(60) + 20(75)}{10 + 20} = 70$$

Your turn: which summary

Your turn

Incomes in a small firm: most near 40, one owner at 400. Which summary best describes a typical income?

- ☐ A. The mean, because it uses every value
- ☐ B. The median, because it is less affected by the single high extreme
- ☐ C. The mode, because it is always best
- ☐ D. Any of them; they are interchangeable

Check: which summary

- 1 Question** Typical income asks for a representative centre, not payroll total per person.
- 2 Median** A single high owner income can pull the mean upward; the median locates the ordered middle.
- 3 Boundary** The mean remains useful for total per person. Exact numerical summaries require all incomes.

Range, IQR, standard deviation

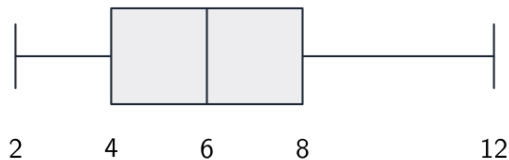
Summary	Calculation and meaning
Range	$R = x_{max} - x_{min}$: full span; directly uses the extremes.
IQR	$IQR = Q_3 - Q_1$: central-half span under the stated convention.
Population spread	$\sigma = \sqrt{\sum (x - \mu)^2 / n}$ for the complete population.
Sample estimate	$s = \sqrt{\sum (x - \bar{x})^2 / (n - 1)}$: use the stated sample convention.

Use the evidence

Variance has squared units; standard deviation has the original units.

Worked example: reading a five-number summary

whiskers: minimum to maximum



- Known: minimum 2, $Q_1 = 4$, median 6, $Q_3 = 8$, maximum 12.
- The overall span and the middle-half span are different quantities.
- The box does not identify every original value, the sample size, or why the spread arose.

$$R = x_{\max} - x_{\min} = 12 - 2 = 10$$
$$IQR = Q_3 - Q_1 = 8 - 4 = 4$$

The normal model and the empirical rule

Interval for $\mu = 50, \sigma = 10$	Approximate probability
$40 < X < 60$ ($\mu \pm \sigma$)	68%
$30 < X < 70$ ($\mu \pm 2\sigma$)	95%
$20 < X < 80$ ($\mu \pm 3\sigma$)	99.7%

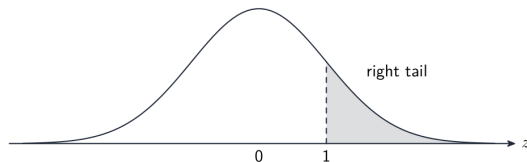
Use the evidence

These are normal-model areas, not guaranteed observed proportions. The mean and SD alone do not prove normality.

z-scores make scales comparable

- 1 **Standardise a threshold** $z = (x - \mu)/\sigma$ measures distance from the mean in standard deviations.
- 2 **Choose the region** $\Phi(z) = P(Z \leq z)$ is the left cumulative area.
- 3 **Use the correct operation** Right tail: $P(Z > z) = 1 - \Phi(z)$. Interval: $P(a < Z < b) = \Phi(b) - \Phi(a)$.
- 4 **Interpret a count** $E(N) = np$ is an expected count, which may be fractional; the actual observed count is an integer.

Worked example: the area above 60



- Known: an invented measurement model has $\mu = 50$ and $\sigma = 10$.
- For values above 60, standardise the threshold: $z = (60 - 50) / 10 = 1$.
- Use the complement of the supplied cumulative area.

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 50}{10} = 1$$

$$P(X > 60) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

Your turn: below 40

Your turn

Same normal model: $\mu = 50$, $\sigma = 10$, $\Phi(1) = 0.8413$. Find $P(X < 40)$.

- A. 0.1587
- B. 0.8413
- C. 0.5000
- D. 0.0228

Check: below 40

- 1 **Standardise** $z = (x - \mu)/\sigma = (40 - 50)/10 = -1$
- 2 **Select the left area** $P(X < 40) = \Phi(-1) = 1 - \Phi(1)$
- 3 **Substitute** $P(X < 40) = 1 - 0.8413 = 0.1587$.

Check yourself

- Rows 4, 7, 10 continuing with the same difference: what are a and d , and what is row 5?
- Why is 20% then 10% off not 30% off?
- Two groups of 10 and 20 with means 60 and 75: why is the combined mean 70, not 67.5?
- Under a normal model with $\mu = 50$, $\sigma = 10$, find z for 60 and identify the region $X > 60$.