

## GAC010 Mathematics II: Probability, Statistics and Finance

GAC010-aligned practice with invented teaching conditions; the centre's brief sets the actual assessment.

## Rules, calculations, and evidence limits

A cheaper headline rate need not produce the cheaper total, and a larger survey need not produce a more representative result.

## The method, every time

- Show the known data, why the formula applies, the symbolic equation and the substitution.
- State the unit and interpret the result.
- A statistical summary describes a defined set of observations or a stated model.
- It does not automatically establish a cause or a whole-population finding.

## Invented teaching conditions

- All financial amounts, rates, fees and pay rules in the examples are invented teaching conditions.
- Use their stated timing and rounding assumptions.
- Do not assume an actual tax or lending rule.

## Two sequences, two growth models

- A sequence is an ordered list of numbers; a series is their sum.
- An arithmetic sequence adds a fixed common difference  $d$  each step.
- A geometric sequence multiplies by a fixed common ratio  $r$ .

$$u_n = a + (n - 1)d \quad \text{and} \quad u_n = ar^{n-1}$$

# The sums, and the sum to infinity

| Series          | Sum, and its condition                                                     |
|-----------------|----------------------------------------------------------------------------|
| Arithmetic      | $S_n = n/2 \text{ times open } 2a + (n-1)d$<br>close                       |
| Geometric       | $S_n = a(1 - r^n)/(1 - r)$ , for $r \neq 1$                                |
| Sum to infinity | $S_{\infty} = a/(1-r)$ , only when the<br>absolute value of $r$ is below 1 |

## Worked example: the fifth row

row 1: 4    •   •   •   •

row 2: 7    •   •   •   •   •   •

row 3: 10   •   •   •   •   •   •   •   •   •

- Known: row counts 4, 7 and 10 continue with the same difference.
- Thus  $a = 4$  and  $d = 3$ .
- For the fifth row, use the arithmetic term formula.

$$u_5 = 4 + (5 - 1)(3) = 16$$

## Worked example: the first five rows together

- 1 **What is asked** The first five rows together, a total — so use the arithmetic sum, not the fifth term.
- 2 **The equation in symbols**  $S_n = n/2 [2a + (n-1)d]$
- 3 **The substitution**  $S_5 = 5/2 [2(4) + (5-1)(3)]$
- 4 **The result, with its meaning**  $S_5 = 50$  dots across five rows.

### Decision

Sixteen was one term; fifty is the sum of five.

$$S_5 = \frac{5}{2}[2(4) + (5 - 1)(3)] = 50$$

## Your turn: term or total

### Your turn

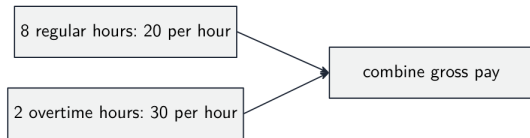
With  $a = 4$  and  $d = 3$ , which formula answers "how many dots in the first 6 rows"?

- A. The term formula  $u_6 = a + 5d$
- B. The sum formula  $S_6 = 6/2 [2a + 5d]$
- C. The geometric sum with  $r = 3$
- D. The sum to infinity

## The vocabulary of pay

| Term              | Meaning                                                                         |
|-------------------|---------------------------------------------------------------------------------|
| Gross pay         | What you earn before deductions                                                 |
| Net pay           | What arrives after deductions such as tax and insurance                         |
| Simple interest   | $I = Prt$ — interest on the original amount only                                |
| Compound interest | $A = P(1+r)^n$ — interest on the interest                                       |
| Depreciation      | A reduction in recorded value; a fixed-percentage model uses $V_n = V_0(1-r)^n$ |

## Worked example: regular and overtime hours



$$G = 8(20) + 2(1.5)(20) = 220 \text{ units};$$

- Known: 8 hours at 20 units per hour, and 2 overtime hours at 1.5 times that rate.
  - Separate the two categories before adding pay.
  - Then take the invented fixed deduction of 15 units once.
- $$N = 220 - 15 = 205 \text{ units}$$

## Successive discounts are multiplications

- A discount of 20% multiplies by 0.8.
- Two successive discounts of 20% and 10% multiply by  $0.8 \times 0.9 = 0.72$ .
- That is a 28% reduction, not 30%.

## Your turn: the pay slip

### Your turn

6 hours at 15 units plus 4 overtime hours at 1.5x, then a 10-unit deduction. What is net pay?

- A. 90 units
- B. 170 units
- C. 180 units
- D. 160 units

## Present value and the effective rate

- Under a stated constant discount rate and timing assumption, the present value of a future amount is  $PV = FV/(1+r)^n$ .
- An annuity has specified payments at regular intervals; a loan schedule may change amounts or adjust a final payment.
- The APR label is a convention — its calculation and included charges depend on the stated convention.

$$PV = \frac{FV}{(1+r)^n}$$

## A nominal rate is not an effective rate

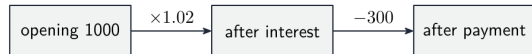
- 1 Read the nominal rate** A nominal 12% teaching rate, divided into monthly periods, gives 1% per month.
- 2 Compound the periods** Effective annual growth is  $(1.01)^{12} - 1$ .
- 3 Compute**  $(1.01)^{12} - 1$  is about 12.68%.
- 4 Respect the boundary** A different compounding frequency or fee schedule needs a different calculation.

### Decision

The monthly compounding lifts 12% to about 12.68% effective.

$$(1.01)^{12} - 1 \approx 12.68\%$$

## Worked example: interest then payment



- Known: opening balance 1000 units, period rate 2%, end-of-period payment 300 units.
- Under the stated convention, interest comes before payment.
- Each new period's interest uses the new balance, not the original principal.

$$B_1 = 1000(1.02) - 300 = 720; \quad B_2 = 720(1.02) - 300 = 434.40$$

## Your turn: the second period

### Your turn

Balance 720 after period 1. Interest 2%, payment 300. What closes period 2?

- A. 434.40 — interest on 720, then the payment
- B. 420.00 — the payment straight off 720
- C. 424.80 — interest on the original 1000
- D. 720.00 — unchanged

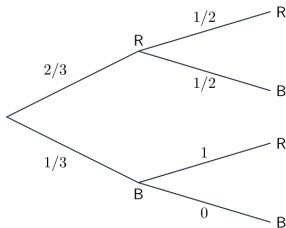
# The probability rulebook

| Rule                      | Statement                                                                               |
|---------------------------|-----------------------------------------------------------------------------------------|
| Equally likely outcomes   | $P(A) = \text{favourable over total}$                                                   |
| Mutually exclusive events | $P(A \text{ or } B) = P(A) + P(B)$                                                      |
| Independent events        | $P(A \text{ and } B) = P(A) \times P(B)$                                                |
| Complement                | $P(\text{not } A) = 1 - P(A)$ — the fast route to "at least one"                        |
| Conditional               | $P(A \text{ given } B) = P(A \text{ and } B) \text{ over } P(B), \text{ for } P(B) > 0$ |

## Use the evidence

A union subtracts its overlap:  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ .

## Worked example: two draws without replacement



- Known: two red and one blue counter; two draws without replacement.
- For two reds, follow the red-red path and multiply.

$$P(RR) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

## Your turn: at least one red

### Your turn

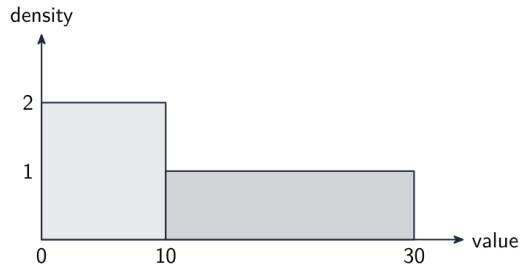
Same bag, two draws without replacement. What is the probability of at least one red?

- A.  $1/3$
- B. 1, by the complement of the impossible blue-blue path
- C.  $2/3$
- D.  $1/6$

## Population, sample, bias

- A population is everyone you want to describe; a sample is who you actually measure.
- Generalising needs a suitably representative sample and an honest account of selection and nonresponse.
- A class survey can describe that class without establishing a school-wide percentage.
- Bias enters through who you ask, who answers, and how the question is worded.

## Worked example: equal areas, unequal heights



$$D_1 = \frac{20}{10} = 2; \quad D_2 = \frac{20}{20} = 1$$

- Known: class 0–10 has frequency 20; class 10–30 has frequency 20.
- Widths differ, so calculate frequency density:  $D = f/w$ .
- The second bar is half as high but twice as wide; both areas represent 20 observations.

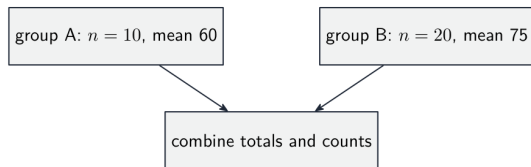
## Mean, median, mode

| Measure    | What it is, and what it costs                                                 |
|------------|-------------------------------------------------------------------------------|
| The mean   | Total over count — uses every value, and every outlier                        |
| The median | The middle value in order — an extreme may move without changing it           |
| The mode   | The most frequent value — works for categories too; may be absent or repeated |

### Use the evidence

A high extreme can pull the mean above the median; neither says most observations equal it.

## Worked example: a combined mean needs totals



- Known: group A has 10 scores with mean 60; group B has 20 with mean 75.
- Recover each group's total before combining.
- The unweighted mean of 60 and 75 would be 67.5, ignoring the unequal counts.

$$\bar{x} = \frac{10(60) + 20(75)}{10 + 20} = \frac{2100}{30} = 70$$

## Your turn: which summary

### Your turn

Incomes in a small firm: most near 40, one owner at 400. Which summary best describes a typical income?

- ☐ A. The mean, because it uses every value
- ☐ B. The median, because the single extreme distorts the mean upward
- ☐ C. The mode, because it is always best
- ☐ D. Any of them; they are interchangeable

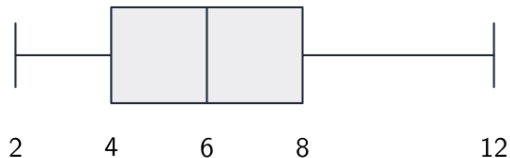
## Range, IQR, standard deviation

- The range is the largest minus the smallest — the full span, so one extreme can change it substantially.
- The interquartile range,  $Q3 - Q1$ , spans the central half by the stated quartile convention.
- A box plot shows quartiles and a median, with whisker rules that must be specified.
- Standard deviation summarises spread using squared deviations; variance has squared units, standard deviation the original units.

$$\sigma = \sqrt{\sum (x - \mu)^2 / n}; \quad s = \sqrt{\sum (x - \bar{x})^2 / (n - 1)}$$

## Worked example: reading a five-number summary

whiskers: minimum to maximum



$$R = 12 - 2 = 10; \quad IQR = 8 - 4 = 4$$

- Known: minimum 2,  $Q1 = 4$ , median 6,  $Q3 = 8$ , maximum 12.
- The overall span and the middle-half span are different quantities.
- The box does not identify every original value, the sample size, or why the spread arose.

## The normal model and the empirical rule

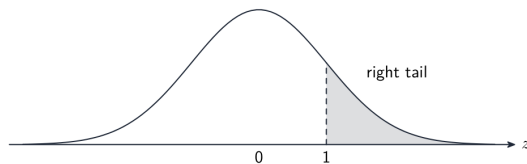
- The normal distribution is a continuous, symmetric, bell-shaped model specified by its mean and positive standard deviation.
- Those two summaries alone do not establish that observed data follow the model.
- Empirical rule: about 68% within one standard deviation, 95% within two, 99.7% within three.

## z-scores make scales comparable

- A z-score says how many standard deviations a value sits from the mean:  $z = (x - \mu)/\sigma$ .
- Use the cumulative area  $\Phi(z) = P(Z \leq z)$ , and symmetry  $\Phi(-z) = 1 - \Phi(z)$ .
- An interval subtracts its two cumulative bounds.
- Expected count is sample size times probability, and need not be an integer; an observed count is an integer.

$$z = \frac{x - \mu}{\sigma}; \quad \Phi(-z) = 1 - \Phi(z)$$

## Worked example: the area above 60



- Known: an invented measurement model has  $\mu = 50$  and  $\sigma = 10$ .
- For values above 60, standardise the threshold:  $z = (60-50)/10 = 1$ .
- Use the complement of the supplied cumulative area.

$$P(X > 60) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

## Your turn: below 40

### Your turn

Same model ( $\mu = 50$ ,  $\sigma = 10$ ). What is  $P(X < 40)$ ?

- A. 0.1587 —  $z = -1$ , and  $\Phi(-1) = 1 - 0.8413$
- B. 0.8413 — the cumulative area at  $z = 1$
- C. 0.5000 — by symmetry about the mean
- D. 0.0228 — the two-standard-deviation tail

## Check yourself

- Rows 4, 7, 10 continuing with the same difference: what are  $a$  and  $d$ , and what is row 5?
- Why is 20% then 10% off not 30% off?
- Two groups of 10 and 20 with means 60 and 75: why is the combined mean 70, not 67.5?
- $\mu = 50$ ,  $\sigma = 10$ : which  $z$  does the threshold 60 give, and which tail?