Course Companion

GAC Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

Contents 目录

1. GAC004 Mathematics I: Fundamentals 2

2. GAC010 Mathematics II: Probability, Statistics and Finance 13

3. GAC016 Mathematics III: Calculus Advanced Applications 25

4. GAC024 Discrete Mathematics 34

GAC004 Mathematics I: Fundamentals

GAC Mathematics

What this module is, and how it is marked

On 16 October 2007 the Shanghai Composite stock index closed at 6,124 points. A year later, on 28 October 2008, it closed at 1,665. That is a fall of about 73%. To get back to 6,124 from 1,665, the index would have had to rise by about 268%, not 73%, because a percentage rise is calculated from the new, smaller value. Percentages, like every topic in Mathematics I, reward a clear method: write what you know, choose the right rule, and check what the answer means.

GAC004 has six units: arithmetic, algebra, graphs, geometry, trigonometry, and exponential and logarithmic functions. Your centre sets the **in-class test** 课堂测验, **projects** 项目, **examination** 考试 and **coursework** 平时作业, with their own weights and deadlines.

In this module you will:

- use the vocabulary of number, percentages, significant figures and standard form;
- expand, factorise and solve linear and quadratic equations, inequalities and simultaneous equations;
- find gradients, midpoints and distances, and read graphs;
- calculate angles, areas and volumes, and use the trigonometric ratios and rules;
- use exponential and logarithmic functions for growth, decay and scales.

Keep a **terminology logbook** 术语记录本: the English term, a meaning in your own words and a small example. Read the **command word** 指令词 in every question: *solve* asks for values, *simplify* for an equivalent expression, *justify* for a reason. Show each step so that a reader can check it.

Terminology and arithmetic review

An **integer** 整数 is a whole number, including negative numbers and zero. A **rational number** 有理数 can be written as a fraction of two integers; an **irrational number** 无理数 cannot, and π and $\sqrt{2}$ are the standard examples. A **prime number** 质数 is a positive integer greater than 1 whose only factors are 1 and itself.

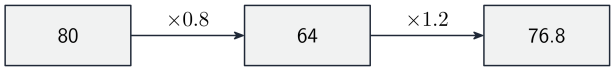
The **order of operations** 运算顺序 fixes what an expression means: brackets, then indices, then multiplication and division, then addition and subtraction, left to right within a level.

$$3 + 4 \times 2^2 = 3 + 4 \times 4 = 3 + 16 = 19$$

- A **factor** 因数 divides a number exactly; a **multiple** 倍数 is the number times a whole number. 6 is a factor of 24, and 24 is a multiple of 6.

- The **highest common factor (HCF)** 最大公因数 and **lowest common multiple (LCM)** 最小公倍数 come from prime factors: $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, so the HCF is $2^2 \times 3 = 12$ and the LCM is $2^3 \times 3^2 = 72$.
- A **percentage** 百分比 is a fraction out of 100. An increase of 15% multiplies by 1.15; a decrease of 15% multiplies by 0.85. To reverse a change, divide by the multiplier.
- **Significant figures** 有效数字 show how precisely a value is given. China's population at the end of 2023 was 1,409,670,000, or 1.41×10^9 to three significant figures.
- **Standard form** 科学记数法 writes a number as $a \times 10^n$ with $1 \le a < 10$ and n an integer. The speed of light is 3.00×10^8 m/s; the mass of an electron is 9.11×10^{-31} kg.

Worked example: a fall and a rise of 20%. A jacket costs ¥80. It is reduced by 20% in a sale, and after the sale the new price is raised by 20%.



each multiplier uses the current amount

¥80 falls to ¥64, then rises to ¥76.80: each percentage uses the price at that stage

$$P_1 = 80 \times 0.8 = ¥64 \qquad P_2 = 64 \times 1.2 = ¥76.80$$

The price is not back at ¥80. The fall was ¥16, but the rise was only ¥12.80, because 20% of 64 is less than 20% of 80. This is the Shanghai index in miniature: a fall needs a bigger percentage rise to recover. The rise needed here is $\dfrac{80}{64} = 1.25$, a 25% increase.

Practice sheet **1.1** gives a progressively harder set with worked solutions.

Algebra I: introductory algebra

In $5x^2 - 3x + 7$, the whole thing is an **expression** 表达式; $5x^2$, $-3x$ and 7 are its **terms** 项; 5 is the **coefficient** 系数 of x^2; and 7 is a **constant** 常数.

An **equation** 方程 says that two expressions are equal and is solved for a value. An **identity** 恒等式 is true for every value. An **inequality** 不等式 compares two expressions with $<$, $\le$, $>$ or $\ge$.

To **expand** 展开 is to remove brackets; to **factorise** 因式分解 is to put them back:

$$(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

- To solve a **linear equation** 一元一次方程, do the same operation to both sides until the letter stands alone.

exercises.lol exercises.com.cn

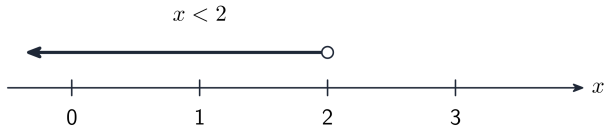
- A **quadratic equation** 一元二次方程 has the form $ax^2 + bx + c = 0$ with $a \neq 0$. Factorise when you can, or use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** 联立方程 are two equations in two unknowns, solved by substitution or elimination. Two straight lines meet at one point, or never (parallel), or everywhere (the same line).
- △ An inequality **reverses** 反向 when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

Worked example: choosing a phone plan. Plan A costs ¥30 a month plus ¥0.10 a minute; plan B costs ¥50 a month with free calls. When is A cheaper?

$$30 + 0.1m < 50 \implies 0.1m < 20 \implies m < 200$$

Plan A is cheaper for fewer than 200 minutes a month; at exactly 200 minutes they cost the same.

Worked example: dividing by a negative. Solve $7 - 3x > 1$.



A number line shows x less than 2, with an open circle at 2.

$$7 - 3x > 1 \implies -3x > -6 \implies x < 2$$

Dividing by -3 reverses the sign. Check: $x = 1$ gives $7 - 3 = 4 > 1$, true; $x = 2$ gives exactly 1, so 2 is excluded (an open circle).

Practice sheet **1.2** gives a progressively harder set with worked solutions.

Algebra: graphs and coordinate geometry

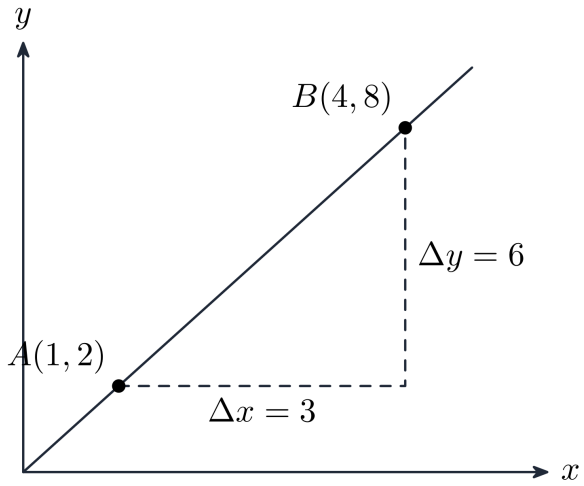
A non-vertical straight line can be written $y = mx + c$, where m is the **gradient** 斜率 and c the **y-intercept** y 轴截距. The gradient is the rise divided by the run:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Lines are **parallel** 平行 when their gradients are equal, and **perpendicular** 垂直 when their gradients multiply to -1.
- The **midpoint** 中点 of a segment is the average of the endpoints; the **distance** 距离 between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

- A quadratic graph is a **parabola** 抛物线. Its **vertex** 顶点 is the turning point; its **roots** 根 are where it meets the x-axis. The path of a thrown ball is close to a parabola.
- To solve two equations **graphically** 用图像求解, draw both and read where they cross; check the answer algebraically, because a drawing is only approximate.

Worked example: a line through two points. Find the line through $A(1,2)$ and $B(4,8)$.



The line passes through A at (1,2) and B at (4,8), with dashed coordinate-change guides.

$$m = \frac{8-2}{4-1} = 2 \qquad c = 2 - 2(1) = 0$$

The line is $y = 2x$: across 3, up 6. In context, if x were kilograms of rice and y its cost in yuan, the gradient would be the price, ¥2 per kilogram, and the intercept 0 would mean no fixed charge.

Gradient in the real world. Road signs give gradients as percentages: a 10% slope rises 10 m for every 100 m across, a gradient of 0.1.

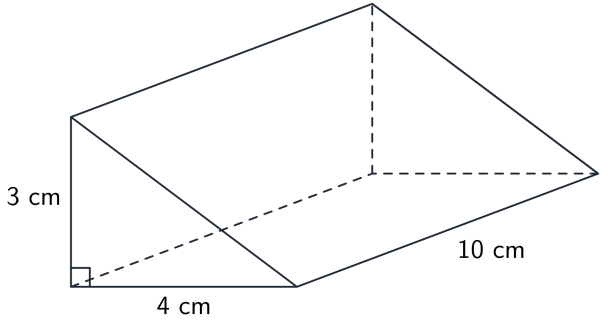
Practice sheet **1.3** gives a progressively harder set with worked solutions.

Geometry: plane, solid and Euclidean geometry

Angles on a straight line add to 180°, angles around a point to 360°, and the angles of a triangle to 180°. In a **polygon** 多边形 with n sides, the interior angles add to $(n-2)\times 180°$; for a hexagon, 720°, which is why each angle of a regular hexagon (like a honeycomb cell) is 120°.

- **Congruent** 全等 shapes are identical in size and shape; **similar** 相似 shapes have the same shape, with all lengths in one ratio.
- **Pythagoras' theorem** 勾股定理: in a right-angled triangle, $a^2 + b^2 = c^2$, where c is the **hypotenuse** 斜边. Ancient Chinese mathematicians knew it as the *gougu* rule; the *Zhoubi Suanjing* shows the 3, 4, 5 triangle.
- **Area** 面积 is measured in square units and **volume** 体积 in cubic units. A cylinder has volume $\pi r^2 h$; a sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.
- △ Scaling is not linear. Double every length of a solid and its area is multiplied by 4, its volume by 8. Galileo used this in 1638 to explain why a giant animal needs much thicker bones: its weight grows with volume, but bone strength grows only with cross-sectional area.

Worked example: a prism. A chocolate box is a triangular prism. Its cross-section is a right-angled triangle with perpendicular sides 3 cm and 4 cm, and the box is 10 cm long.



not to scale

A triangular prism has a right-triangle cross-section with perpendicular sides 3 and 4 cm and length 10 cm; perspective is not to scale.

$$A = \tfrac{1}{2}(3)(4) = 6 \text{ cm}^2 \qquad V = A \times L = 6 \times 10 = 60 \text{ cm}^3$$

Use the perpendicular sides for the area, not the sloping side (which is 5 cm, by Pythagoras).

Practice sheet **1.4** gives a progressively harder set with worked solutions.

exercises.lol exercises.com.cn

Trigonometry



The London Eye is 135 m tall; the height of a capsule rises and falls like a sine curve

In a right-angled triangle, with θ one of the acute angles:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

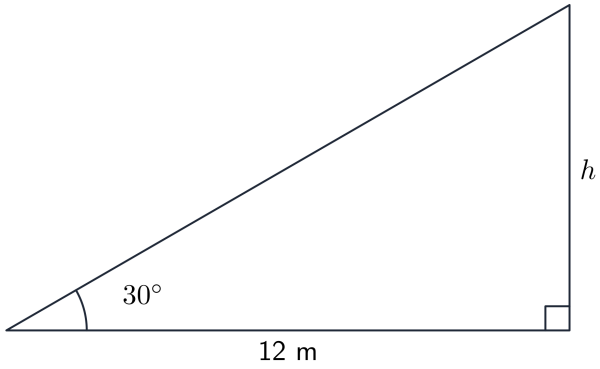
For any triangle, the **sine rule** 正弦定理 and the **cosine rule** 余弦定理 apply:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad a^2 = b^2 + c^2 - 2bc\cos A$$

- An angle can be measured in **degrees** 度 or **radians** 弧度; π radians $= 180°$. Check your calculator's mode before every question.
- The **unit circle** 单位圆 extends sine and cosine beyond 90°; they repeat every 360°, so they are **periodic** 周期的.
- A **bearing** 方位角 is measured clockwise from north and written with three figures, such as 075°.
- **Angles of elevation and depression** 仰角与俯角 are measured from the horizontal, up and down. Add the observer's eye height if the question asks for height above the ground.

> **Worked example: the height of a tree.** From a point 12 m from the foot of a tree, the angle of elevation of the top is 30°.

exercises.lol exercises.com.cn



A right triangle has a 12 m horizontal base and an angle of elevation of 30 degrees; the unknown height is opposite the angle.

$$h = 12\tan 30° \approx 6.93 \text{ m}$$

This is the height above eye level. If the observer's eyes are 1.6 m above the ground, the tree is about $6.93 + 1.6 = 8.53$ m tall.

Trigonometry at planet scale. Around 240 BCE, Eratosthenes noticed that at noon on midsummer the Sun shone straight down a well at Syene, while at Alexandria, about 5,000 stadia to the north, it was 7.2° from overhead. Since 7.2° is $\frac{1}{50}$ of a full circle, the Earth's circumference must be about $50 \times 5000 = 250{,}000$ stadia, close to the modern value.

Practice sheet **1.5** gives a progressively harder set with worked solutions.

Exponential and logarithmic functions



A seismograph: earthquake magnitude is a logarithmic scale, so each step up is ten times the shaking

An **exponential function** 指数函数 has the form $y = a^x$ with $a > 0$ and $a \neq 1$: the variable is in the **exponent** 指数. Each step of 1 in x multiplies y by a. It grows when $a > 1$ and decays when $0 < a < 1$; its graph passes through $(0, 1)$ and never reaches the x-axis.

A **logarithm** 对数 answers the reverse question: $\log_a y = x$ means $a^x = y$, so $\log_{10} 1000 = 3$. Logarithms are defined only for positive numbers.

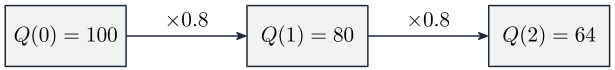
$$\log(mn) = \log m + \log n, \qquad \log\left(\frac{m}{n}\right) = \log m - \log n, \qquad \log(m^k) = k \log m$$

- The laws turn multiplication into addition, which solves equations with the unknown in the exponent.
- **Natural logarithms** 自然对数 use the base $e \approx 2.718$ and are written ln.
- **Compound interest** 复利 is exponential: $A = P(1 + r)^n$.
- **Exponential decay** 指数衰减 has a base between 0 and 1. Carbon-14 has a half-life of about 5,730 years, so after t years a fraction $\left(\frac{1}{2}\right)^{t/5730}$ remains; this is how archaeologists date wood and bone.

Logarithmic scales. Earthquake magnitude is logarithmic: each step of 1 means ten times the ground movement on a seismograph (and about 32 times the energy). The Wenchuan earthquake of 12 May 2008 had magnitude 8.0; a magnitude 6.0 earthquake produces one hundredth of its ground movement, because $10^{8-6} = 100$.

> **Worked example: decay by a fixed percentage.** A new electric car loses 20% of its value each year. It costs ¥100,000.

exercises.lol exercises.com.cn



equal time periods, changing absolute reductions

A model starting at 100 is multiplied by 0.8 each period, giving 80 then 64.

$$V(n) = 100000 \times 0.8^n \qquad V(2) = 100000 \times 0.8^2 = ¥64{,}000$$

The first year's loss is ¥20,000, the second year's only ¥16,000: the same percentage of a smaller value. When is the car worth less than half its price? Solve $0.8^n < 0.5$:

$$n > \frac{\log 0.5}{\log 0.8} = \frac{-0.3010}{-0.0969} \approx 3.1$$

so after 4 whole years. (Dividing by $\log 0.8$, a negative number, reverses the inequality.) Practice sheet **1.6** gives a progressively harder set with worked solutions.

Sources for this lesson

- GAC004 Mathematics I course outline.
- Shanghai Stock Exchange, Composite Index closing values, 16 October 2007 and 28 October 2008.
- National Bureau of Statistics of China, population at the end of 2023.
- Galileo Galilei, *Two New Sciences* (1638), on scaling and the strength of bones.
- Cleomedes, *On the Heavens*, the ancient account of Eratosthenes' measurement.
- China Earthquake Administration, magnitude of the 2008 Wenchuan earthquake.

Common errors

- Undoing a percentage change with the same percentage: a 20% fall then a 20% rise does not return to the start.
- Multiplying a minus sign into only the first term of a bracket: $-2(x-4) = -2x+8$.
- Keeping the inequality sign when multiplying or dividing by a negative number.
- Swapping rise and run, or mixing the order of the coordinates, in a gradient.
- Mixing area and volume units, or radius and diameter, in mensuration formulas.
- Using SOH CAH TOA in a triangle without a right angle; use the sine or cosine rule there.
- Leaving the calculator in radian mode for a degree question.
- Treating percentage growth as adding a fixed amount each period.

Key terms

- **integer**: a whole number, positive, negative or zero.
- **irrational number**: a number that cannot be written as a fraction of two integers.
- **significant figures**: the digits that show how precisely a number is given.
- **standard form**: a number written as $a \times 10^n$ with $1 \le a < 10$.
- **coefficient**: the number multiplying a variable in a term.
- **identity**: an equation that is true for every value of the variable.
- **gradient**: the steepness of a line, rise divided by run.
- **similar**: having the same shape, with all lengths in the same ratio.
- **hypotenuse**: the longest side of a right-angled triangle, opposite the right angle.
- **bearing**: a direction measured clockwise from north, in three figures.
- **logarithm**: the power to which a base must be raised to give a number.
- **exponential decay**: a decrease by the same percentage in each period.

Check yourself

Answer each question before reading the answers.

1. A price of ¥80 rises by 25% and the new price then falls by 20%. Find the final price and explain the result.
2. Simplify $3(2x - 5) - 2(x - 4)$.
3. Solve $5 - 3x > 11$.
4. Solve the simultaneous equations $2x + y = 7$ and $x - y = 2$.
5. Solve (a) $x^2 - 5x + 6 = 0$; (b) $2x^2 + 3x - 2 = 0$.
6. A line passes through $(1, 3)$ and $(4, 9)$. Find (a) its gradient; (b) its equation; (c) where it crosses the x-axis; (d) the gradient of a perpendicular line.
7. A closed cylindrical can has radius 3 cm and height 10 cm. Find its volume and total surface area, to 1 decimal place.
8. A 5 m ladder makes an angle of $70°$ with level ground. How high up the wall does it reach, and how far is its foot from the wall?
9. (a) In triangle ABC, $a = 8$, $B = 40°$ and $C = 75°$. Find b. (b) Two sides of a triangle are 7 and 9 with an included angle of $50°$. Find the third side.
10. A town of 5,000 people grows by 3% a year. Find the population after 10 years, and the number of whole years until it first doubles.
11. Evaluate $\log_2 8 + \log_2 4 - \log_2 2$, and solve $2^x = 20$ to 2 decimal places.

Answers: 1. $80 \times 1.25 = 100$, then $100 \times 0.8 = 80$: back to ¥80. The 20% is taken from the larger amount, ¥100, so it removes ¥20, the same ¥20 that was added. 2. $6x - 15 - 2x + 8 = 4x - 7$. 3. $-3x > 6$, so $x < -2$ (dividing by -3 reverses the sign). 4. Adding gives $3x = 9$, so $x = 3$ and $y = 1$; check: $2(3) + 1 = 7$ and $3 - 1 = 2$. 5. (a) $(x - 2)(x - 3) = 0$, so $x = 2$ or 3; (b) $(2x - 1)(x + 2) = 0$, so $x = \tfrac{1}{2}$ or $x = -2$. 6. (a) $\dfrac{9 - 3}{4 - 1} = 2$; (b) $y = 2x + 1$; (c) $x = -\tfrac{1}{2}$, the point $\left(-\tfrac{1}{2}, 0\right)$; (d) $-\tfrac{1}{2}$. 7. $V = 90\pi \approx 282.7$ cm^3; surface area $= 18\pi + 60\pi = 78\pi \approx 245.0$ cm^2. 8. Height $= 5\sin 70° \approx 4.70$ m; distance $= 5\cos 70° \approx 1.71$ m. 9. (a) $A = 65°$, $b = \dfrac{8\sin 40°}{\sin 65°} \approx 5.67$; (b) $c^2 = 49 + 81 - 126\cos 50° \approx 49.01$, so $c \approx 7.00$. 10. $5000 \times 1.03^{10} \approx 6720$; doubling needs $n \ge \dfrac{\ln 2}{\ln 1.03} \approx 23.4$, so 24 whole years. 11. $3 + 2 - 1 = 4$; $x = \dfrac{\log 20}{\log 2} \approx 4.32$.

exercises.lol exercises.com.cn

GAC010 Mathematics II: Probability, Statistics and Finance

GAC Mathematics

What this module is, and how it is marked

In 1936 the American magazine *The Literary Digest* posted ten million ballots and received about 2.4 million back. From them it predicted that Alf Landon would beat President Franklin Roosevelt easily. Roosevelt won 46 of the 48 states. George Gallup, with a sample of about 50,000 chosen to match the voting population, called the result correctly. The magazine's names had come largely from telephone directories and car registrations, which in the Depression meant wealthier voters. A bigger sample was worse, because it was the wrong sample. Mathematics II is about questions like this: which formula, which model, and what the numbers can and cannot tell you.

GAC010 covers sequences, finance, probability and statistics. Your centre sets the **test** 测验, **projects** 项目, **examination** 考试 and **coursework** 平时作业, with their own weights and deadlines.

In this module you will:

- use arithmetic and geometric sequences and their sums;
- calculate pay, deductions, discounts, interest, depreciation and loan balances;
- find probabilities with tables, trees and counting;
- collect, display and summarise data, and choose the right average and spread;
- use the normal distribution and z-scores.

In every calculation, write what you know, the formula and why it applies, the substitution, and the answer with its unit. Then say what the answer means.

Sequences and series

A **sequence** 数列 is an ordered list of numbers; a **series** 级数 is their sum.

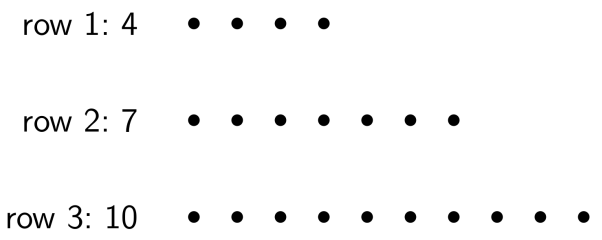
An **arithmetic sequence** 等差数列 adds a fixed **common difference** 公差 d each step. A **geometric sequence** 等比数列 multiplies by a fixed **common ratio** 公比 r.

$$u_n = a + (n-1)d \qquad \text{and} \qquad u_n = ar^{\,n-1}$$

- The sums are $S_n = \frac{n}{2}\,(2a + (n-1)d)$ and $S_n = \dfrac{a(1-r^n)}{1-r}$ for $r \neq 1$.
- A geometric series with $|r| < 1$ has a **sum to infinity** 无穷和: $S_\infty = \dfrac{a}{1-r}$.
- Keep a term separate from a total: u_n is one value, S_n is the sum of the first n.

Where the arithmetic sum comes from. A famous story, told by his first biographer, says that the young Carl Friedrich Gauss was set to add $1 + 2 + \cdots + 100$ at school and answered at once. He paired the numbers from the two ends: $1 + 100$, $2 + 99$, and so on, 50 pairs each making 101, so the sum is $50 \times 101 = 5050$. The formula $S_n = \frac{n}{2}(\text{first} + \text{last})$ is the same idea.

> **Worked example 1: rows of seats.** A small theatre has 4 seats in the front row, 7 in the next, then 10, and so on.



Rows of 4, 7 and 10: a constant difference of 3

Known: $a = 4$, $d = 3$. The fifth row has

$$u_5 = 4 + (5-1)(3) = 16 \text{ seats}$$

The first five rows together need the sum, not the fifth term:

$$S_5 = \frac{5}{2}\,[2(4) + (5-1)(3)] = \frac{5}{2}(20) = 50 \text{ seats}$$

> **Worked example 2: a bouncing ball.** A ball is dropped from 2 m and each bounce reaches 60% of the previous height. The bounce heights form a geometric sequence with $a = 1.2$ m and $r = 0.6$. The total height of all the bounces, if they went on for ever, is $S_\infty = \dfrac{1.2}{1-0.6} = 3$ m.

Practice sheet **2.1** gives a progressively harder set with worked solutions.

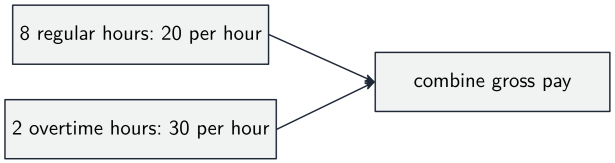
Finance: earning money and purchasing goods

- **Gross pay** 税前工资 is what you earn; **net pay** 税后工资 is what arrives after **deductions** 扣除项 such as tax and social insurance.
- **Simple interest** 单利 is $I = Prt$: interest on the original amount only.
- **Compound interest** is $A = P(1+r)^n$: interest is added to the balance and then earns interest itself, so the balances form a geometric sequence.
- **Depreciation** 折旧 is a fall in value. A fixed-percentage model uses $V_n = V_0(1-r)^n$.

exercises.lol exercises.com.cn

- A **discount** 折扣 of 20% multiplies by 0.8. Two successive discounts of 20% and 10% multiply by $0.8 \times 0.9 = 0.72$, which is a 28% reduction, not 30%.

> **Worked example: overtime pay.** China's Labour Law (Article 44) says that overtime on a normal working day must be paid at least 150% of the normal wage, on a rest day 200% (if no time off is given in return), and on a public holiday 300%.



A pay record: 8 regular hours and 2 overtime hours at 1.5 times the rate

A worker earns ¥20 an hour, works 8 normal hours and 2 hours of weekday overtime, and has ¥15 deducted. Separate the two kinds of hours before adding:

$$G = 8(20) + 2(1.5)(20) = 160 + 60 = ¥220$$

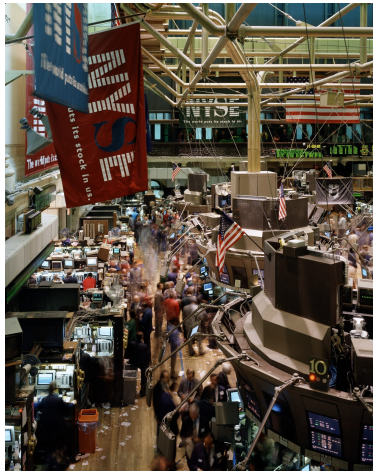
$$N = G - D = 220 - 15 = ¥205$$

> **Worked example: a discount on a phone.** A phone costs ¥6,000. In a sale it is 20% off, and members get a further 10% off the sale price. The price paid is $6000 \times 0.8 \times 0.9 = ¥4{,}320$, a total discount of ¥1,680, or 28%.

Practice sheet **2.2** gives a progressively harder set with worked solutions.

Finance: investing money and loans

- The **present value** 现值 of a future amount, at a constant rate r per period, is $PV = \dfrac{FV}{(1+r)^n}$: how much you would need to invest now to have FV later.
- An **annuity** 年金 is a series of equal payments at regular intervals.
- In a **loan repayment** 贷款还款 schedule, each period interest is added to the balance and the payment is taken off. The order matters.
- An **APR** 年利率 is an annual rate. If interest is added monthly, the rate per month is the APR divided by 12, and the effective yearly growth is higher: a nominal 12% with 1% a month gives $(1.01)^{12} - 1 \approx 12.68\%$.



A stock exchange: investments grow or shrink by percentages, so compound interest formulas apply

> **Worked example: a loan balance.** A ¥1,000 loan charges 2% interest per month, added at the end of each month, after which a payment of ¥300 is made.



One month of a loan: ¥1,000 plus 2% interest, then a ¥300 payment

$$B_1 = 1000(1.02) - 300 = ¥720$$

The second month's interest is on ¥720, not on the original ¥1,000:

$$B_2 = 720(1.02) - 300 = ¥434.40$$

> **Worked example: present value.** How much must be invested now at 4% a year, compounded yearly, to have ¥50,000 for university fees in 5 years?

$$PV = \frac{50000}{1.04^5} = \frac{50000}{1.21665} \approx ¥41{,}096.36$$

Practice sheet **2.3** gives a progressively harder set with worked solutions.

exercises.lol exercises.com.cn

Probability: key concepts

Probability 概率 measures how likely an event is, on a scale from 0 to 1.



Two dice: list the 36 equally likely outcomes before counting the ones you want

- For equally likely outcomes, $P(A) = \dfrac{\text{favourable outcomes}}{\text{total outcomes}}$.
- Events are **mutually exclusive** 互斥 when they cannot both happen: $P(A \text{ or } B) = P(A) + P(B)$.
- Events are **independent** 独立 when one does not affect the other: $P(A \text{ and } B) = P(A) \times P(B)$.
- The **complement** 对立事件 rule, $P(\text{not } A) = 1 - P(A)$, is often the fastest route to an answer containing "at least one".
- **Counting techniques** 计数方法 give the number of outcomes: a **permutation** 排列 counts ordered selections, a **combination** 组合 counts unordered ones.
- For events that overlap, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; conditional probability is $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$.

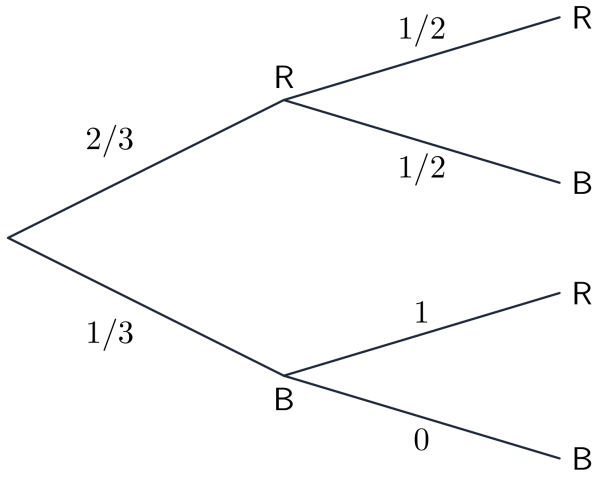
A surprising result: the birthday problem. In a group of 23 people, the probability that at least two share a birthday is just over 50%. Use the complement: the probability that all 23 birthdays are different is

$$\frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{343}{365} \approx 0.493$$

so the probability of at least one shared birthday is $1 - 0.493 = 0.507$. Most people guess far lower, because they think of matches with themselves, not between all 253 pairs.

exercises.lol exercises.com.cn

Worked example: drawing without replacement. A bag holds two red counters and one blue. Two are drawn without putting the first back.



A two-draw tree for a bag with two red and one blue counter, without replacement, labels conditional second-draw probabilities.

$$P(RR) = P(R_1) \times P(R_2 \mid R_1) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

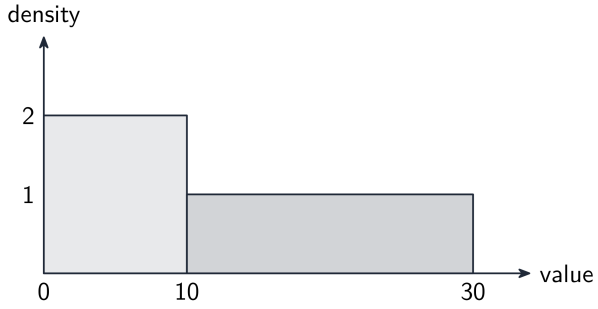
If the first counter is blue, only red is left, so blue-blue is impossible. The draws are not independent: the first changes the second.

Practice sheet **2.4** gives a progressively harder set with worked solutions.

Statistics: collecting and displaying data

- A **population** 总体 is everyone you want to describe; a **sample** 样本 is who you actually measure.
- To generalise, the sample must be **representative** 有代表性的: chosen so that every part of the population has a fair chance of being included.
- **Qualitative data** 定性数据 are categories; **quantitative data** 定量数据 measure amounts, either **discrete** 离散 (counted) or **continuous** 连续 (measured).
- **Bias** 偏差 enters through who you ask, who answers and how the question is worded. The *Literary Digest* poll had all three problems: its lists favoured the wealthy, and the 24% who replied were a self-selected group.

Worked example: a histogram with unequal classes. Bar *area* shows frequency, so the height is the frequency density.



A histogram has intervals 0 to 10 and 10 to 30 with densities 2 and 1; their areas each represent frequency 20.

Class 0 to 10 has frequency 20; class 10 to 30 has frequency 20.

$$D_1 = \frac{20}{10} = 2 \qquad D_2 = \frac{20}{20} = 1$$

The second bar is half as high but twice as wide; both areas represent 20 observations. Use separated bars for categories and touching bars (a histogram) for continuous intervals.

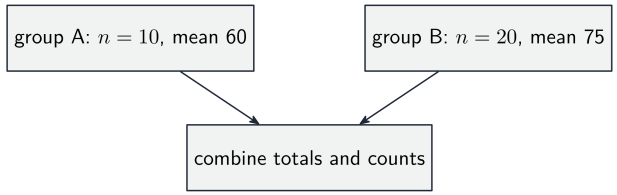
Practice sheet **2.5** gives a progressively harder set with worked solutions.

Statistics: measures of central location

- The **mean** 平均数 is the total divided by the count. It uses every value, including extreme ones.
- The **median** 中位数 is the middle value when the data are in order. Extreme values do not move it.
- The **mode** 众数 is the most frequent value; it also works for categories.

Mean or median? A real example. China's National Bureau of Statistics reports both. In 2023 the mean disposable income per person was about ¥39,218, but the median was about ¥33,036. A small number of high incomes pull the mean up; the median shows what a person in the middle actually has. For skewed data such as income or house prices, the median is usually the better "typical" value.

> **Worked example: combining two groups.** Class A has 10 students with a mean score of 60; class B has 20 students with a mean of 75.



Two groups contribute different totals: 10 learners with mean 60, and 20 learners with mean 75.

Recover each class's total first:

$$\bar{x} = \frac{10(60) + 20(75)}{10 + 20} = \frac{2100}{30} = 70$$

The simple average of 60 and 75 would be 67.5, which ignores the fact that class B is twice as large.

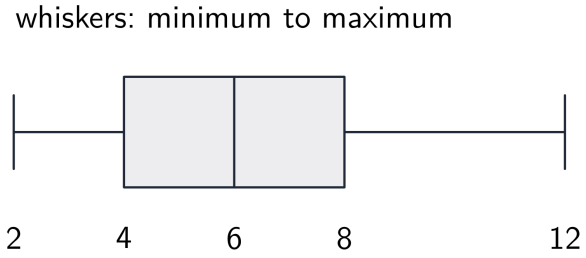
Practice sheet **2.6** gives a progressively harder set with worked solutions.

Statistics: measures of variability

- The **range** 全距 is the largest value minus the smallest, so one extreme value can change it a lot.
- The **interquartile range (IQR)** 四分位距 is $Q_3 - Q_1$, the spread of the middle half. A **box plot** 箱线图 shows the minimum, quartiles, median and maximum.
- The **standard deviation** 标准差 measures spread using squared distances from the mean. For a whole population, $\sigma = \sqrt{\sum(x-\mu)^2/n}$; for a sample estimate, $s = \sqrt{\sum(x-\bar{x})^2/(n-1)}$. It has the same units as the data.
- Two data sets can share a mean and be completely different: a city with every day at 20 °C and one swinging between 0 °C and 40 °C have the same mean temperature.

> **Worked example: a box plot.** The five-number summary is minimum 2, $Q_1 = 4$, median 6, $Q_3 = 8$, maximum 12.

exercises.lol exercises.com.cn



A minimum-to-maximum box plot marks minimum 2, first quartile 4, median 6, third quartile 8 and maximum 12.

$$\text{range} = 12 - 2 = 10 \qquad IQR = 8 - 4 = 4$$

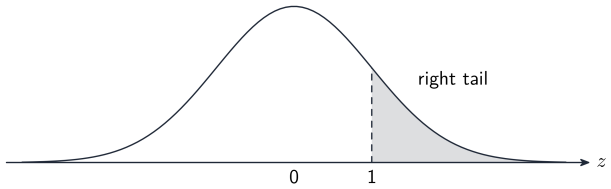
The long right whisker shows that the top quarter of the data is spread more widely than the bottom quarter.

Practice sheet **2.7** gives a progressively harder set with worked solutions. On sheet 2.7, quartiles are the medians of the lower and upper halves (leaving out the overall median when the count is odd), and the whiskers run to the minimum and maximum.

Distributions: the normal distribution

- The **normal distribution** 正态分布 is a symmetric, bell-shaped model described by its mean μ and standard deviation σ. Many measurements, such as adult heights, are close to normal.
- About 68% of values lie within one standard deviation of the mean, 95% within two, and 99.7% within three.
- A **z-score** 标准分 says how many standard deviations a value is from the mean: $z = \dfrac{x - \mu}{\sigma}$. It lets you compare values on different scales.
- Use the cumulative area $\Phi(z) = P(Z \le z)$ and symmetry: $\Phi(-z) = 1 - \Phi(z)$.

Worked example: a test score. Scores on a large test are modelled as normal with $\mu = 50$ and $\sigma = 10$. What proportion score above 60?



A standard-normal curve marks z equal to 1 and shades the right-tail region beyond it.

$$z = \frac{60 - 50}{10} = 1 \qquad P(X > 60) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

About 16% of candidates score above 60. In a group of 400, we would expect about $400 \times 0.1587 \approx 63$ of them to do so.

Comparing with z-scores. Li scores 72 in maths ($\mu = 60$, $\sigma = 8$) and 80 in English ($\mu = 70$, $\sigma = 10$). Her z-scores are 1.5 and 1.0, so her maths result is the stronger one, even though the raw mark is lower.

Practice sheet **2.8** gives a progressively harder set with worked solutions.

Sources for this lesson

- GAC010 Mathematics II course outline (sequences, finance, probability and statistics).
- *Labour Law of the People's Republic of China* (1994), Article 44, on overtime pay.
- National Bureau of Statistics of China, *Statistical Communiqué on the 2023 National Economic and Social Development* (per capita disposable income, mean and median).
- P. Squire, "Why the 1936 Literary Digest Poll Failed", *Public Opinion Quarterly* 52 (1988).
- W. Sartorius von Waltershausen, *Gauss zum Gedächtniss* (1856), the source of the Gauss story.

Common errors

- Using n instead of $n - 1$ in $a + (n - 1)d$ or ar^{n-1}.
- Giving a term when the question asks for a sum, or the reverse.
- Applying simple interest where the question compounds, or the reverse.
- Adding successive percentage discounts: 20% then 10% off is 28% off, not 30%.
- Taking off a loan payment before adding interest when the schedule says interest first.
- Multiplying probabilities of dependent events as if they were independent; forgetting to change the denominator when drawing without replacement.
- Thinking a bigger sample is always better: a biased sample stays biased however large it is.
- Finding the median without first putting the data in order.
- Mixing the population ($\div n$) and sample ($\div (n - 1)$) standard deviations.

exercises.lol exercises.com.cn

Key terms

- **sequence**: an ordered list of numbers.
- **series**: the sum of the terms of a sequence.
- **common difference**: the fixed amount added each step in an arithmetic sequence.
- **common ratio**: the fixed number multiplied each step in a geometric sequence.
- **net pay**: pay after deductions such as tax.
- **present value**: the amount needed now to reach a future amount at a given rate.
- **mutually exclusive**: describes events that cannot happen together.
- **independent**: describes events where one does not change the probability of the other.
- **representative**: describes a sample that fairly reflects the population.
- **median**: the middle value of ordered data.
- **interquartile range**: the spread of the middle half of the data, $Q_3 - Q_1$.
- **standard deviation**: a measure of how far values typically lie from the mean.
- **z-score**: the number of standard deviations a value lies from the mean.

Check yourself

Answer each question before reading the answers.

1. For the arithmetic sequence $7, 11, 15, \ldots$, find the 20th term and the sum of the first 20 terms.
2. For the geometric sequence $3, 6, 12, \ldots$, find the 8th term and the sum of the first 8 terms.
3. A worker earns ¥25 an hour for 40 normal hours and works 4 hours of overtime on a rest day at 200% (no time off given). Deductions are 10% of gross pay. Find the gross and net pay.
4. ¥2,000 is invested for 3 years at 5% a year. Find the final amount with (a) simple interest; (b) annual compound interest.
5. A car worth ¥240,000 loses 15% of its value each year. Find its value after 3 years, to the nearest yuan.
6. Two fair dice are thrown. Find P(sum is 7) and P(at least one six).
7. For the data $4, 7, 7, 9, 10, 12, 14$, find the mean, median, mode, range and IQR (quartiles as medians of the half-lists, excluding the overall median).
8. Find the population standard deviation of $2, 4, 4, 4, 5, 5, 7, 9$.
9. Heights are modelled as normal with mean 170 cm and standard deviation 8 cm. Find (a) P(height < 186 cm); (b) the approximate percentage between 162 and 178 cm; (c) P(height > 180 cm).

Answers: 1. $u_{20} = 7 + 19(4) = 83$; $S_{20} = \frac{20}{2}(7 + 83) = 900$. 2. $u_8 = 3(2)^7 = 384$; $S_8 = \dfrac{3(2^8 - 1)}{2 - 1} = 765$. 3. Gross $= 40(25) + 4(2)(25) = 1000 + 200 =$ ¥1,200; net $= 1200 \times 0.9 =$ ¥1,080. 4. (a) $I = 2000(0.05)(3) = 300$, total ¥2,300; (b) $2000(1.05)^3 =$ ¥2,315.25. 5. $240000(0.85)^3 = 240000(0.614125) =$ ¥147,390. 6. $\frac{6}{36} = \frac{1}{6}$; $1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$. 7. Mean $= \frac{63}{7} = 9$; median 9; mode 7; range $14 - 4 = 10$; lower half $4, 7, 7$ gives $Q_1 = 7$, upper half $10, 12, 14$ gives $Q_3 = 12$, so IQR $= 5$. 8. Mean $= 40/8 = 5$; squared deviations $9, 1, 1, 1, 0, 0, 4, 16$ sum to 32; variance $= 32/8 = 4$; $\sigma = 2$. 9. (a) $z = \frac{186 - 170}{8} = 2$, so $P \approx 0.9772$; (b) 162 to 178 cm is within one standard deviation of the mean, so about 68%; (c) $z = 1.25$, so $P = 1 - 0.8944 = 0.1056$.

exercises.lol exercises.com.cn

GAC016 Mathematics III: Calculus & Advanced Applications

GAC Mathematics

What this module is, and how it is marked

At the 2009 World Championships in Berlin, Usain Bolt ran 100 metres in 9.58 seconds. His average speed was $100 \div 9.58 \approx 10.4$ m/s. But he was not running at 10.4 m/s at any fixed moment: he started from rest, and timing data showed him passing 12 m/s between 60 and 80 metres. An average speed is a gradient between two points; the speed at one instant is the gradient of a tangent. Calculus, developed independently by Isaac Newton and Gottfried Leibniz in the late seventeenth century, is the mathematics of that difference.

GAC016 develops **differentiation** 微分, **integration** 积分 and their applications. Your centre sets the assessment tasks and their weights.

In this module you will:

- find derivatives from first principles and with the power, product, quotient and chain rules;
- find and classify stationary points;
- integrate, find definite integrals and areas, and use substitution;
- solve optimisation, connected-rates and area-between-curves problems.

State the variable, the domain and the units in every answer. $\dfrac{dy}{dx}$ is the rate of change of y with respect to x; in an application, say what the rate means. A volume changing at -3 cm^3/s is shrinking by 3 cm^3 every second; the volume itself is not negative.

Differentiation

A **derivative** 导数 is a rate of change: how fast y changes as x changes. Geometrically it is the **gradient of the tangent** 切线斜率 at a point.

exercises.lol exercises.com.cn



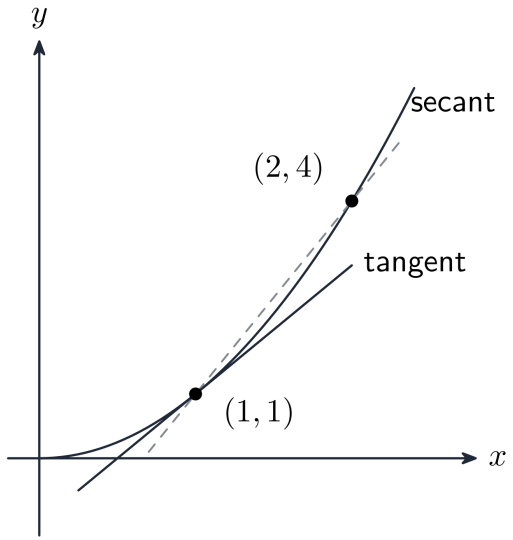
A roller coaster: velocity is the derivative of position, and the steepest point of the track is where the gradient is largest

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

That limit is the definition. **Differentiation from first principles** 从定义求导 works out the gradient of a short chord, from x to $x + h$, and then lets h shrink to zero. A function with a sharp corner, such as $|x|$ at $x = 0$, has no derivative there: the gradients from the left and the right do not agree.

- The **power rule** 幂法则: if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$.
- The **product rule** 乘积法则: $(uv)' = u'v + uv'$.
- The **quotient rule** 商法则, where $v \neq 0$: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$.
- The **chain rule** 链式法则: if $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x)$.
- **Stationary points** 驻点 occur where $f'(x) = 0$. The **second derivative** 二阶导数 then classifies it: negative gives a local **maximum** 极大值, positive a local **minimum** 极小值. If the second derivative is zero, check the sign of $f'(x)$ on each side; $y = x^3$ at $x = 0$ is a stationary point that is neither.

> **Worked example.** a tangent slope differs from a secant slope



The curve y equals x squared has a tangent at (1,1) and a dashed secant joining (1,1) to (2,4).

Known: $f(x) = x^2$. The secant between x equal to 1 and 2 has slope 3, but the derivative at x equal to 1 is 2.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

$$m_{tan} = f'(1) = 2(1) = 2$$

The average gradient over an interval is not the gradient at its start: Bolt's average speed was not his speed at any chosen moment.

Worked example. the chain and product rules together

Differentiate $y = x^2(3x - 1)^4$.
Let $u = x^2$ and $v = (3x-1)^4$. Then $u' = 2x$ and, by the chain rule, $v' = 4(3x-1)^3 \cdot 3 = 12(3x - 1)^3$.

$$\frac{dy}{dx} = 2x(3x - 1)^4 + 12x^2(3x - 1)^3 = 2x(3x - 1)^3\big[(3x - 1) + 6x\big] = 2x(3x - 1)^3(9x - 1)$$

Taking out the common factor $2x(3x - 1)^3$ makes the stationary points easy to read: $x = 0$, $x = \frac{1}{3}$ and $x = \frac{1}{9}$.

Worked example. find and classify stationary points

For $f(x) = x^3 - 6x^2 + 9x + 2$, $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so the stationary points are at $x = 1$ and $x = 3$.
$f''(x) = 6x - 12$. At $x = 1$, $f''(1) = -6 < 0$: a local maximum at $(1, 6)$. At $x = 3$, $f''(3) = 6 > 0$: a local minimum at $(3, 2)$.
Always give both coordinates: substitute back into f, not into f'.
Practice sheet **3.1** gives a progressively harder set with worked solutions.

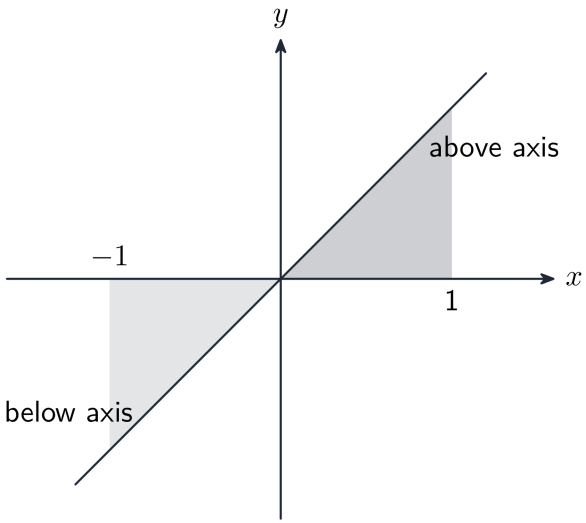
Integration

Integration reverses differentiation. If velocity is the derivative of position, then integrating velocity over a time interval gives the change in position. The **fundamental theorem of calculus** 微积分基本定理 says that a definite integral equals the difference of an antiderivative at its two limits. Area below the axis counts as negative, so a definite integral is a signed total, not always a geometric area.

- An **indefinite integral** 不定积分 has no limits and needs the **constant of integration** 积分常数: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** 定积分 has limits and gives a number: $\int_a^b f(x) \, dx = F(b) - F(a)$.
- **Integration by substitution** 换元积分法 reverses the chain rule.
- △ Forgetting $+c$ on an indefinite integral loses marks on answers that are otherwise correct.

Worked example. negative and positive areas can cancel

exercises.lol exercises.com.cn



The line y equals x crosses the axis at zero, with shaded regions on both sides over negative 1 to 1.

Known: $f(x) = x$ from negative 1 to 1. Its definite integral is zero because the negative and positive contributions cancel.

$$I = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

Geometric area instead adds the magnitudes of the two triangular regions.

$$A = -\int_{-1}^{0} x\,dx + \int_{0}^{1} x\,dx = \frac{1}{2} + \frac{1}{2} = 1$$

Worked example. integration by substitution

Find $\int 6x(x^2+1)^5\,dx$.

Let $u = x^2 + 1$, so $\dfrac{du}{dx} = 2x$ and $6x\,dx = 3\,du$.

$$\int 6x(x^2+1)^5\,dx = \int 3u^5\,du = \frac{u^6}{2} + c = \frac{(x^2+1)^6}{2} + c$$

Check by differentiating: $\dfrac{d}{dx}\dfrac{(x^2+1)^6}{2} = 3(x^2+1)^5 \cdot 2x = 6x(x^2+1)^5$. ✓

For a definite integral, either change the limits to u-values or substitute back before using the original x-limits; never mix them.

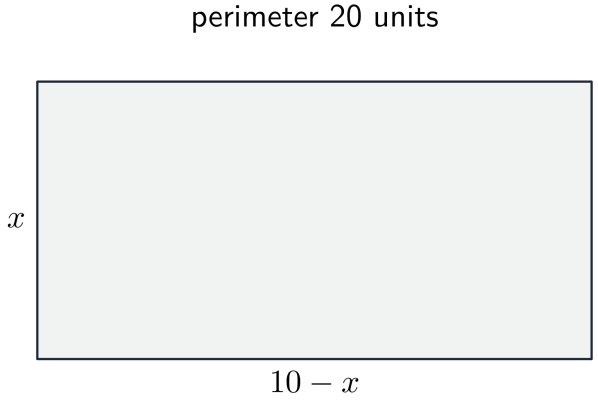
exercises.lol exercises.com.cn

Practice sheet **3.2** gives a progressively harder set with worked solutions.

Advanced applications

- **Optimisation** 最优化 finds the largest or smallest value of a quantity: write the quantity as a function of one variable, state the domain, differentiate, find the stationary points, and check the ends of the domain.
- **Rates of change** 变化率 chain together: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$.
- **Area between two curves** 两曲线间面积 is $\displaystyle\int_a^b (f(x) - g(x))\,dx$ where f is the upper curve on that whole interval. Split at crossings if their order changes.
- **Marginal cost and revenue** 边际成本与边际收益 are the derivatives of cost and revenue: approximately the extra cost or revenue of one more unit.

Worked example. a fixed perimeter leaves one area variable



A rectangle has width x and length 10 minus x under a fixed perimeter of 20 units.

Known: a rectangle has perimeter 20, so length plus width is 10. Let width be x; length is $10 - x$ and $0 < x < 10$.

$$A(x) = x(10 - x)$$

$$A'(x) = 10 - 2x$$

The stationary candidate is $x = 5$. Since $A''(x) = -2 < 0$ and the area approaches zero at both domain ends, this candidate is the global maximum. It is a 5 by 5 square with area 25.

Practice sheet **3.3** gives a progressively harder set with worked solutions.

Worked example. connected rates of change

A spherical balloon is inflated at $\dfrac{dV}{dt} = 50$ cubic centimetres per second. Find how fast the radius grows when $r = 5$ cm.

$V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dr} = 4\pi r^2 = 100\pi$ at $r = 5$.

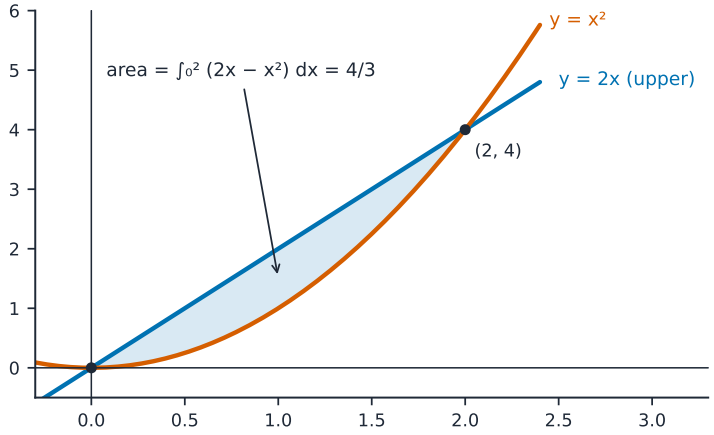
$$\frac{dr}{dt} = \frac{dV/dt}{dV/dr} = \frac{50}{100\pi} = \frac{1}{2\pi} \approx 0.159 \text{ cm per second}$$

The same inflation rate gives a slower radius change as the balloon grows, because each extra centimetre of radius needs more volume.

Worked example. area between two curves

Find the area enclosed by $y = x^2$ and $y = 2x$.

They meet where $x^2 = 2x$, so $x = 0$ and $x = 2$. On $0 < x < 2$, the line is above the parabola (at $x = 1$: $2 > 1$).



Upper curve minus lower curve, between the crossings: 4/3

$$A = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

exercises.lol exercises.com.cn

Sources for this lesson

- GAC016 Mathematics III course outline.
- IAAF (now World Athletics), biomechanics report on the 2009 Berlin World Championships 100 m final.
- I. Newton, *Philosophiae Naturalis Principia Mathematica* (1687); G. W. Leibniz, "Nova Methodus pro Maximis et Minimis" (1684).

Common errors

- Using the secant slope when the question asks for an instantaneous rate, or the reverse.
- Forgetting the inner derivative in the chain rule: $\dfrac{d}{dx}(3x-1)^4 \neq 4(3x-1)^3$.
- Writing $(uv)' = u'v'$. The product rule has two terms.
- Swapping the numerator order in the quotient rule: it is $u'v - uv'$, not $uv' - u'v$.
- Classifying a stationary point with $f'' = 0$ as a point of inflection without checking the sign of f' on both sides.
- Dropping $+c$, or treating a definite integral of a sign-changing function as an area.
- Not checking the ends of the feasible domain in an optimisation question.
- Leaving the answer to a related-rates question without units or a sign that shows increase or decrease.

Key terms

- **derivative**: the rate of change of a function; the gradient of its tangent.
- **first principles**: finding a derivative from the limit of chord gradients.
- **chain rule**: the rule for differentiating a function of a function.
- **stationary point**: a point where the derivative is zero.
- **second derivative**: the derivative of the derivative, used to classify stationary points.
- **indefinite integral**: an antiderivative, written with $+c$.
- **definite integral**: an integral with limits, giving a signed total.
- **optimisation**: finding the largest or smallest value of a quantity.
- **marginal cost**: the derivative of cost; about the cost of one more unit.

Check yourself

Answer each question before reading the answers.

1. Differentiate (a) $y = 5x^4 - \dfrac{3}{x} + 2\sqrt{x}$; (b) $y = (2x+3)^5$; (c) $y = x\,e^{2x}$; (d) $y = \dfrac{x^2}{x+1}$.

2. Use first principles to show that the derivative of $f(x) = 3x^2$ is $6x$.
3. Find the equation of the tangent to $y = x^3 - 2x$ at $x = 2$.
4. Find and classify the stationary points of $y = 2x^3 - 3x^2 - 12x + 5$.
5. Find (a) $\displaystyle\int (4x^3 - 6x + 1)\,dx$; (b) $\displaystyle\int_1^4 \sqrt{x}\,dx$; (c) $\displaystyle\int 2x(x^2-3)^4\,dx$.
6. Evaluate $\displaystyle\int_0^3 (x-2)\,dx$, then find the total area between $y = x-2$ and the x-axis for $0 \le x \le 3$. Explain why the two answers differ.
7. An open box is made from a 12 cm by 12 cm square sheet by cutting a square of side x cm from each corner. Show that $V = x(12-2x)^2$, state the domain and find the maximum volume.
8. A firm's cost is $C(q) = 0.02q^2 + 4q + 500$ yuan for q units. Find the marginal cost at $q = 100$ and explain what it means.
9. The radius of a circular oil spill increases at 0.5 m per minute. Find the rate of increase of its area when $r = 20$ m.
10. Find the area enclosed by $y = 4 - x^2$ and $y = x + 2$.

Answers: 1. (a) $20x^3 + \dfrac{3}{x^2} + \dfrac{1}{\sqrt{x}}$. (b) $10(2x+3)^4$. (c) $e^{2x}(1+2x)$. (d) $\dfrac{2x(x+1)-x^2}{(x+1)^2} = \dfrac{x^2+2x}{(x+1)^2}$. 2. $\dfrac{3(x+h)^2-3x^2}{h} = \dfrac{6xh+3h^2}{h} = 6x+3h$ for $h \neq 0$; as $h \to 0$ this tends to $6x$. 3. At $x=2$, $y=4$ and $\dfrac{dy}{dx} = 3x^2 - 2 = 10$. Tangent: $y - 4 = 10(x-2)$, so $y = 10x - 16$. 4. $y' = 6x^2 - 6x - 12 = 6(x-2)(x+1)$. $y'' = 12x - 6$. At $x=-1$: $y=12$, $y''=-18<0$, local maximum $(-1, 12)$. At $x=2$: $y=-15$, $y''=18>0$, local minimum $(2,-15)$. 5. (a) $x^4 - 3x^2 + x + c$. (b) $\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$. (c) With $u = x^2-3$: $\dfrac{(x^2-3)^5}{5} + c$. 6. $\left[\frac{x^2}{2} - 2x\right]_0^3 = \frac{9}{2} - 6 = -\frac{3}{2}$. The line is below the axis on $0 < x < 2$ (area 2) and above on $2 < x < 3$ (area $\frac{1}{2}$), so the total area is $\frac{5}{2}$. The integral counts the part below the axis as negative. 7. The base is $(12-2x)$ by $(12-2x)$ and the height is x; the domain is $0 < x < 6$. $V' = (12-2x)^2 - 4x(12-2x) = (12-2x)(12-6x)$, so $x = 2$ (as $x = 6$ is excluded). $V(2) = 2 \cdot 8^2 = 128$ cm^3; $V \to 0$ at both ends, so this is the maximum. 8. $C'(q) = 0.04q + 4$, so $C'(100) = 8$ yuan per unit: near $q = 100$, one more unit costs about ¥8 more. 9. $\dfrac{dA}{dt} = 2\pi r \cdot \dfrac{dr}{dt} = 2\pi(20)(0.5) = 20\pi \approx 62.8$ m^2 per minute. 10. $4 - x^2 = x + 2$ gives $x^2 + x - 2 = 0$, so $x = -2$ or $x = 1$. The parabola is above on this interval: $\displaystyle\int_{-2}^{1} (2 - x - x^2)\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{9}{2}$.

exercises.lol exercises.com.cn

GAC024 Discrete Mathematics

GAC Mathematics

What this module is, and how it is marked

In 1736 the people of Königsberg asked Leonhard Euler a puzzle: could anyone walk through their city crossing each of its seven bridges exactly once? Euler replaced the land with points and the bridges with lines, and proved that no such walk exists: a walk that uses every line once can have at most two points with an odd number of lines, and Königsberg had four. With that proof he started graph theory, the mathematics behind metro maps, satellite navigation and the internet. Discrete mathematics is the study of separate, countable things: sets, digits, logic, step-by-step procedures and networks.

GAC024 covers sets, counting systems, binary logic, algorithms and networks. Your centre sets the assessment tasks, their weights and the tools you may use.

In this module you will:

- use sets, Venn diagrams, relations and functions;
- convert between decimal, binary and hexadecimal;
- use Boolean logic, truth tables and logic gates;
- trace algorithms and compare their efficiency;
- model problems with graphs and find shortest paths.

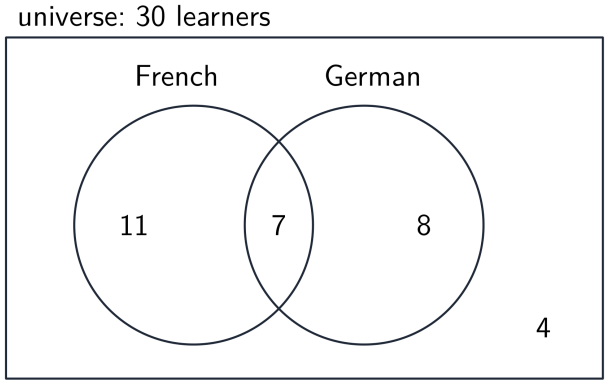
Before solving, state the universe of a set problem, the number of bits in a binary one, or the edge weights in a network.

Sets, relations and functions

- A **set** 集合 is a collection of distinct objects. Order and repetition do not matter.
- **Union** 并集 $A \cup B$ is everything in either; **intersection** 交集 $A \cap B$ is what is in both; the **set complement** 补集 is everything in the stated universe but outside the set.
- A **subset** 子集 has all its elements inside another set.
- A **relation** 关系 pairs elements of two sets. A **function** 函数 is a relation in which each input has exactly one output. "Each student → their birthday" is a function (two students may share a birthday); "each birthday → the students born on it" is not, because one input can have several outputs.
- A **Venn diagram** 韦恩图, introduced by the English logician John Venn in 1880, turns a set problem into a picture. Check that the regions add up to the total.

The **inclusion-exclusion principle** 容斥原理 subtracts the twice-counted overlap once: $|A \cup B| = |A| + |B| - |A \cap B|$.

Worked example. subtract an overlap only once



A class universe of 30 is divided into French-only 11, both 7, German-only 8 and neither 4.

Known: 30 learners, 18 study French, 15 German, and 7 both. The overlap is included in both subject totals.

$$|F \cup G| = |F| + |G| - |F \cap G| = 18 + 15 - 7 = 26$$

$$N_{neither} = |U| - |F \cup G| = 30 - 26 = 4$$

French-only is $18 - 7 = 11$ and German-only $15 - 7 = 8$. The four disjoint regions sum to 30.

Practice sheet **4.1** gives a progressively harder set with worked solutions.

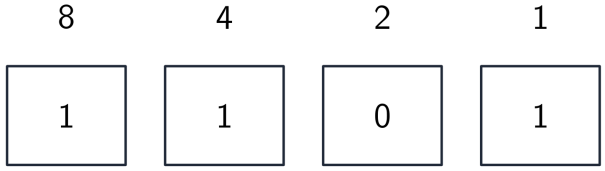
Counting systems

- A **number base** 进制 b uses the digits 0 to $b-1$, and each place is worth a power of b. **Decimal** 十进制 uses ten, **binary** 二进制 two, **hexadecimal** 十六进制 sixteen (digits 0–9 then A–F).
- Every digit's value is its **place value** 位值: in binary the places are 1, 2, 4, 8, 16 and so on.
- Hexadecimal is shorthand for binary: one hex digit is exactly four bits. Web colours use it: `#FF8000` means red 255, green 128, blue 0, a bright orange.

With n bits you can store the whole numbers 0 to $2^n - 1$. An IPv4 internet address has 32 bits, so there are $2^{32} \approx 4.3$ billion addresses, which is why the world ran out of them in 2011 and moved to 128-bit IPv6. If a sum needs more bits than are available, the extra carry is lost (overflow).

exercises.lol exercises.com.cn

Worked example. place weights determine the decimal value



place weights above binary digits

The binary digits 1101 are aligned with place weights 8, 4, 2 and 1.

Known numeral 1101_2. Use weights from right to left: 1, 2, 4 and 8.

$$V = \sum d_i 2^i$$

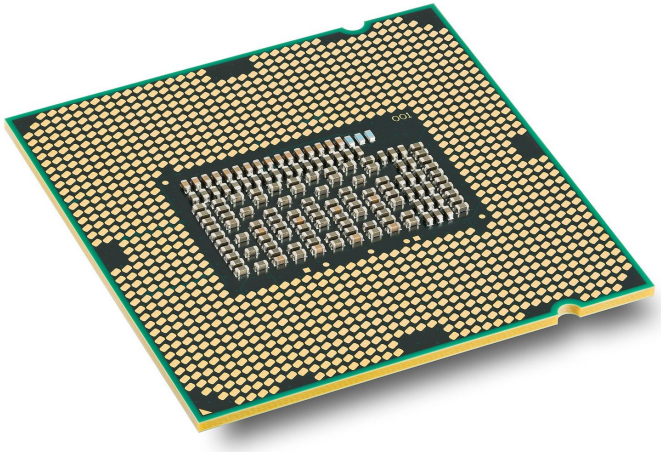
$$V = 1(8) + 1(4) + 0(2) + 1(1) = 13$$

The same value is D in hexadecimal, because $13 = 8 + 4 + 1$ and D stands for 13.

Practice sheet **4.2** gives a progressively harder set with worked solutions.

Binary applications

- **Binary arithmetic** 二进制运算 adds like decimal, carrying at 2 instead of at 10.
- A **bit** 位 is one binary digit; a **byte** 字节 is eight.
- **Boolean algebra** 布尔代数 works on true and false with AND, OR and NOT.
- A **truth table** 真值表 lists every combination of inputs and the output for each. Two expressions are equivalent if their truth tables match in every row.
- **Logic gates** 逻辑门 carry out these operations in electronics, and a **logic circuit** 逻辑电路 connects them. George Boole published his algebra of logic in 1854; in 1937 Claude Shannon showed that it describes switching circuits exactly, which is the basis of every digital computer.



A processor works only in binary: every number, letter and instruction is stored as 0s and 1s

OR is inclusive: it is true when either or both inputs are true. **De Morgan's laws** give $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ and $\neg(A \vee B) = (\neg A) \wedge (\neg B)$: negating a bracket flips each part and swaps AND with OR.

Worked example. an OR output is inverted by NOT



Inputs A and B enter an OR-labelled block, whose output enters a NOT-labelled block to give Y.

Known: $Y = \neg(A \vee B)$. Inclusive OR is false only when both inputs are false; NOT reverses that result. In row order $(A, B) = (0,0), (0,1), (1,0), (1,1)$, the output column is 1, 0, 0, 0. De Morgan gives equivalent expression $(\neg A) \wedge (\neg B)$.

Practice sheet **4.3** gives a progressively harder set with worked solutions.

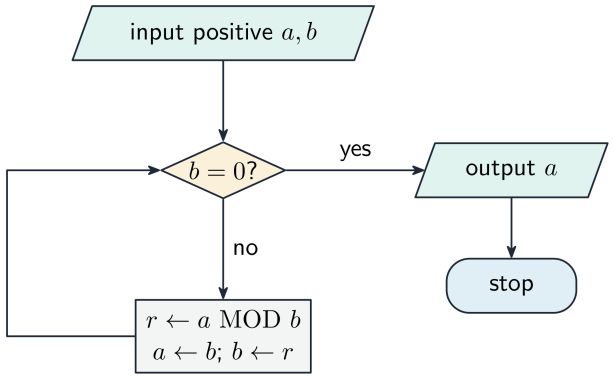
Algorithms

- An **algorithm** 算法 is a list of clear steps that solves a task and always finishes. The word comes from the name of al-Khwarizmi, a ninth-century mathematician in Baghdad.
- A **flowchart** 流程图 draws it: a decision is a diamond, a process a rectangle.
- **Pseudocode** 伪代码 writes the steps in structured English, without a particular programming language.

exercises.lol exercises.com.cn

- **Tracing** 追踪 an algorithm means following it by hand in a table with one column per variable and one row per step.
- **Efficiency** 效率 matters. A linear search checks items one by one, up to n checks. A binary search of a sorted list halves the range each time: a sorted list of 1,000,000 names needs at most 20 checks, because $2^{20} > 1{,}000{,}000$.

Worked example. repeat a remainder step until the second number is zero



A Euclidean-algorithm flowchart tests b equal to zero, otherwise computes a remainder and updates the pair before returning to the test.

This is Euclid's algorithm for the highest common factor, written down around 300 BCE and still used in computer cryptography. Start with $a = 10$ and $b = 6$. While $b \neq 0$: set $r = a \bmod b$, then $a = b$ and $b = r$.

step	a	b	$r = a \bmod b$
1	10	6	4
2	6	4	2
3	4	2	0
end	2	0	

The output is $a = 2$, the HCF of 10 and 6. The algorithm always stops, because each remainder is smaller than the one before.

Practice sheet **4.4** gives a progressively harder set with worked solutions.

Graphs and networks

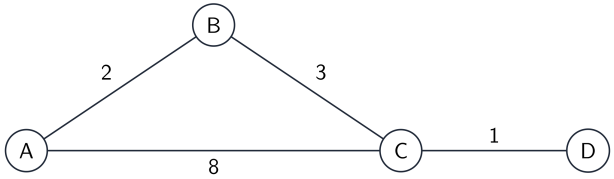
- A **graph** 图 is a set of **vertices** 顶点 joined by **edges** 边. It models anything with connections: roads, friendships, dependencies.

- The **degree** 度 of a vertex is the number of edges that meet it. Every edge has two ends, so the degrees always add up to twice the number of edges. This is the fact behind Euler's Königsberg proof.
- A **tree** 树 is a connected graph with no cycles; a tree with n vertices has $n - 1$ edges. A family tree and a computer's folder structure are trees.
- A **shortest path** 最短路径 problem asks for the route with the smallest total weight (distance, time or cost), not the fewest edges. Edsger Dijkstra designed his famous algorithm for it in 1956; navigation apps still use versions of it.
- A **minimum spanning tree** connects every vertex with the smallest total edge weight and no cycles, the cheapest way to link towns with cable.



A metro map is a network: stations are vertices and the lines between them are edges

Worked example. compare total route weight, not the number of edges



An undirected network joins A to B with weight 2, B to C with 3, A to C with 8 and C to D with 1.

Known edge weights are AB = 2, BC = 3, AC = 8 and CD = 1. The path A-C-D has weight 9, while A-B-C-D has weight 6. So the path with three edges is shorter than the path with two. The minimum spanning tree here uses AB, BC and CD, total 6; it happens to contain the shortest path, but in general the two problems have different

exercises.lol exercises.com.cn

answers.

Practice sheet **4.5** gives a progressively harder set with worked solutions.

Sources for this lesson

- GAC024 Discrete Mathematics course outline.
- L. Euler, "Solutio problematis ad geometriam situs pertinentis" (1736), the Königsberg bridges.
- G. Boole, *An Investigation of the Laws of Thought* (1854); C. E. Shannon, "A Symbolic Analysis of Relay and Switching Circuits" (1938).
- E. W. Dijkstra, "A Note on Two Problems in Connexion with Graphs", *Numerische Mathematik* 1 (1959).
- Euclid, *Elements*, Book VII, Propositions 1–2.

Common errors

- Adding the two set totals without subtracting the overlap: $|A \cup B| = |A| + |B| - |A \cap B|$.
- Calling a relation a function when one input has two outputs, or rejecting one because two inputs share an output.
- Reading binary place values from the left: the rightmost bit is worth 1.
- Grouping binary into fours from the left when converting to hexadecimal; group from the right.
- Forgetting that a fixed-width result keeps only the low bits when the question states wraparound.
- Treating OR as exclusive: inclusive OR is true when both inputs are true.
- Changing only one part when applying De Morgan: $\neg(A \vee B) = \neg A \wedge \neg B$; the operator flips too.
- Choosing the path with fewest edges instead of the smallest total weight.

Key terms

- **set**: a collection of distinct objects.
- **intersection**: the elements that are in both sets.
- **function**: a relation in which each input has exactly one output.
- **place value**: what a digit is worth because of its position.
- **truth table**: a table of every input combination and its output.
- **algorithm**: a finite list of clear steps that solves a task.
- **tracing**: following an algorithm by hand, step by step.
- **degree**: the number of edges meeting at a vertex.
- **tree**: a connected graph with no cycles.

exercises.lol exercises.com.cn

- **shortest path**: the route between two vertices with the smallest total weight.

Check yourself

Answer each question before reading the answers.

1. In a group of 40 students, 25 play football, 18 play basketball and 8 play both. How many play neither? How many play football only?
2. $U = \{1, 2, \ldots, 10\}$, $A = \{1, 2, 3, 4, 6\}$ and $B = \{2, 4, 6, 8\}$. List $A \cup B$, $A \cap B$ and A'.
3. Is $\{(1, a), (2, b), (3, a)\}$ a function from $\{1, 2, 3\}$? Is its inverse relation a function? Explain.
4. Convert 45 to binary and to hexadecimal, and convert $3\mathrm{F}_{16}$ to decimal.
5. Add $1011_2 + 0110_2$. Give the result in full and as a 4-bit stored result with wraparound.
6. What is the largest unsigned value an 8-bit byte can hold, and how many different values can it hold?
7. Complete a truth table for $A \wedge \neg B$.
8. Use De Morgan's law to simplify $\neg(\neg A \vee B)$.
9. Trace the Euclidean algorithm (while $b \neq 0$: $r = a \bmod b$, $a = b$, $b = r$) for $a = 84$, $b = 36$.
10. A sorted list holds 100 items. Find the worst-case number of comparisons for a linear search and for a binary search.
11. A simple graph has vertex degrees $3, 3, 2, 2, 2$. How many edges does it have? How many edges does a tree with 12 vertices have?
12. A network has edges $AB = 4$, $AC = 2$, $BC = 1$, $BD = 5$, $CD = 8$, $DE = 3$, $CE = 10$. Find the shortest path from A to E, and a minimum spanning tree with its total weight.

Answers: 1. $25 + 18 - 8 = 35$ play at least one, so 5 play neither. Football only: $25 - 8 = 17$. 2. $A \cup B = \{1, 2, 3, 4, 6, 8\}$; $A \cap B = \{2, 4, 6\}$; $A' = \{5, 7, 8, 9, 10\}$. 3. Yes: each input has exactly one output. The inverse is not a function, because a would have two outputs, 1 and 3. 4. $45 = 32 + 8 + 4 + 1 = 101101_2$. Grouping from the right: $0010\,1101 = 2\mathrm{D}_{16}$. $3\mathrm{F}_{16} = 3(16) + 15 = 63$. 5. $11 + 6 = 17 = 10001_2$. Keeping the low 4 bits gives 0001_2. 6. $2^8 - 1 = 255$; there are $2^8 = 256$ values, from 0 to 255. 7. $(A, B) = (0, 0) \to 0$; $(0, 1) \to 0$; $(1, 0) \to 1$; $(1, 1) \to 0$. 8. $\neg(\neg A \vee B) = \neg\neg A \wedge \neg B = A \wedge \neg B$. 9. $(84, 36)$: $r = 12$, pair $(36, 12)$; $r = 0$, pair $(12, 0)$. Output 12. 10. Linear: 100 in the worst case. Binary: each comparison halves the range, so at most 7 (since $2^7 = 128 > 100$). 11. Degree sum $= 12 = 2\times$ edges, so 6 edges. A tree with 12 vertices has 11 edges. 12. Shortest $A \to E$: A–C–B–D–E, weight $2 + 1 + 5 + 3 = 11$ (compare A–C–$E = 12$ and A–B–D–$E = 12$). MST by smallest edges without cycles: $BC = 1$, $AC = 2$, $DE = 3$, skip AB (cycle), $BD = 5$: total 11, with 4 edges for 5 vertices.