

Course Companion

GAC Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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GAC004 Mathematics I: Fundamentals

GAC Mathematics

What this module is, and how it is marked

A price can fall by 20% and then rise by 20% without returning to its starting value. Mathematics I helps you explain such results with a method, units and a clear interpretation. The six units connect arithmetic, algebra, graphs, geometry, trigonometry and exponential models.

Your centre's current assessment brief is the authority for tasks, weights, permitted tools and deadlines. An **in-class test** 课堂测验 is completed under stated classroom conditions. **Projects** 项目 can involve collecting data, calculating and reporting; an **examination** 考试 and **coursework** 平时作业 may have different instructions. Do not infer an official assessment pattern from these practice sheets.

A **terminology logbook** 术语记录本 can record the English term, a meaning in your own words and a small example. Follow your centre's instructions if it is submitted or assessed.

Read the **command word** 指令词: *solve* asks for values satisfying a condition, *simplify* for an equivalent expression, and *justify* for a reason. Show a valid method so a reader can check how the result follows. The mark scheme for each task determines its marks.

The six accompanying practice sheets are original GAC004-aligned material, not official papers. Their solutions award marks for stated mathematical steps rather than for the length of a written report.

Terminology and arithmetic review

The vocabulary of number is assumed by every later unit.

An **integer** 整数 is a whole number, including negative whole numbers and zero. A **rational number** 有理数 can be written as a fraction of two integers with a nonzero denominator; an **irrational number** 无理数 cannot, and π and $\sqrt{2}$ are the standard examples. A **prime number** 质数 is a positive integer greater than 1 with exactly two positive factors, itself and 1.

The **order of operations** 运算顺序 fixes what a written expression means: brackets, then indices, then multiplication and division, then addition and subtraction, working left to right within a level.

$$3 + 4 \times 2^2 = 3 + 4 \times 4 = 3 + 16 = 19$$

- A **factor** 因数 divides a number exactly; a **multiple** 倍数 is what you get by multiplying it. 6 is a factor of 24, and 24 is a multiple of 6.
- The **highest common factor (HCF)** 最大公因数 and **lowest common multiple (LCM)** 最小公倍数 come from the prime factors: $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, so the HCF is $2^2 \times 3 = 12$ and the LCM is $2^3 \times 3^2 = 72$.
- A **percentage** 百分比 is a fraction with denominator 100. An increase of 15% multiplies by 1.15; a decrease of 15% multiplies by 0.85. Reversing a percentage change means dividing, never subtracting the same percentage back.
- **Significant figures** 有效数字 are the digits retained to express a value at a stated precision. They do not alone establish the accuracy of a measurement. 0.004 072 to three significant figures is 0.004 07.
- **Standard form** 科学记数法 writes a number as $a \times 10^n$ with $1 \leq |a| < 10$ for a nonzero number, with integer n. It is how a calculator shows a very large or very small answer.

Worked example. a decrease and increase use different bases



each multiplier uses the current amount

Known: an invented price of 80 units falls by 20%, then rises by 20%. Why use successive multipliers? Each change is calculated from the current price.

$$P_{new} = P_{old}(1 - r/100)$$

$$P_1 = 80(1 - 20/100) = 64$$

$$P_{new} = P_{old}(1 + r/100)$$

$$P_2 = 64(1 + 20/100) = 76.8$$

The price is not back at 80. The decrease was 16, while the later increase was 12.8, because the bases differ.

Continue with practice sheet **1.1**. Solve before opening the solutions, and check both the method and the final units.

Algebra I: introductory algebra

Algebra is arithmetic with letters standing for numbers, and it has its own vocabulary.

In $5x^2 - 3x + 7$, the whole thing is an **expression** 表达式; $5x^2$, $-3x$ and 7 are its **terms** 项; 5 is the **coefficient** 系数 of x^2 ; and 7 is a **constant** 常数.

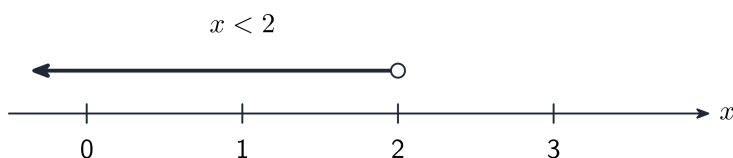
An **equation** 方程 says two expressions are equal and is solved for a value. An **identity** 恒等式 is true for every value in its domain. An **inequality** 不等式 compares two expressions with $<$, $\leq$, $>$ or $\geq$.

To **expand** 展开 is to remove brackets; to **factorise** 因式分解 is to put them back.

$$(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

- Solving a **linear equation** 一元一次方程 means doing the same operation to both sides until the letter stands alone.
- A **quadratic equation** 一元二次方程 has the form $ax^2 + bx + c = 0$ with $a \neq 0$. Factorise when suitable, or use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** 联立方程 are two equations in two unknowns, solved by substitution or by elimination. For two linear equations, a unique solution is their intersection. Parallel distinct lines have no solution; identical lines have infinitely many.
- $\triangle$ An inequality **reverses** 反向 when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

Worked example. reverse the inequality sign when dividing by a negative



Known: solve $7 - 3x > 1$. Subtract 7 from both sides, then divide by negative 3. The last operation reverses the inequality.

$$7 - 3x > 1 \implies -3x > -6$$

$$x < \frac{-6}{-3} = 2$$

The open circle excludes 2. Check $x = 1$: $7 - 3(1) = 4 > 1$; the boundary $x = 2$ gives equality and is excluded.

Continue with practice sheet **1.2**. Solve before opening the solutions, and check both the method and the final units.

Algebra: graphs and coordinate geometry

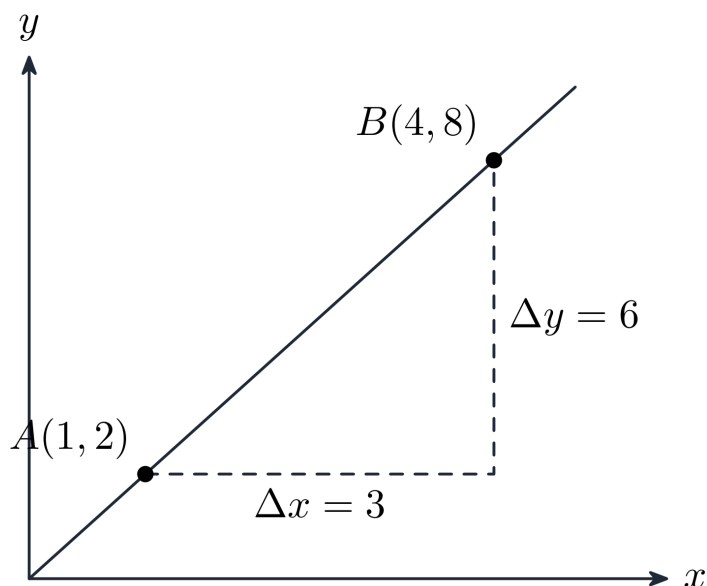
A graph turns an equation into a picture, and coordinate geometry measures that picture.

A nonvertical straight line can be written $y = mx + c$, where m is the **gradient** 斜率 and c the **y-intercept** y 轴截距. The gradient is the rise divided by the run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Distinct nonvertical lines are **parallel** 平行 when their gradients are equal. Two nonvertical lines are **perpendicular** 垂直 when their gradients multiply to -1 . Vertical lines have undefined gradient; a vertical and a horizontal line are perpendicular.
- The **midpoint** 中点 of a segment is the average of the endpoints, and the **distance** 距离 between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graphs as a **parabola** 抛物线. Its **vertex** 顶点 is the turning point, and the **roots** 根 are the x -values where it meets the x -axis, the solutions of $y = 0$. A repeated root touches the axis without crossing; some quadratics have no real roots.
- To solve two equations **graphically** 用图像求解, draw both and read the coordinates of the intersection. The answer is only as accurate as the drawing, which is why an algebraic check matters.

Worked example. a rising line through two given points



Known: $A = (1, 2)$ and $B = (4, 8)$. Use gradient because the line is nonvertical.

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{8 - 2}{4 - 1} = 2$$

The line equation is $y = mx + c$. Substitute A to find the intercept.

$$c = y_A - mx_A = 2 - 2(1) = 0$$

Thus $y = 2x$. The diagram's horizontal change is 3 and vertical change is 6.

Continue with practice sheet **1.3**. Solve before opening the solutions, and check both the method and the final units.

Geometry: plane, solid and Euclidean geometry

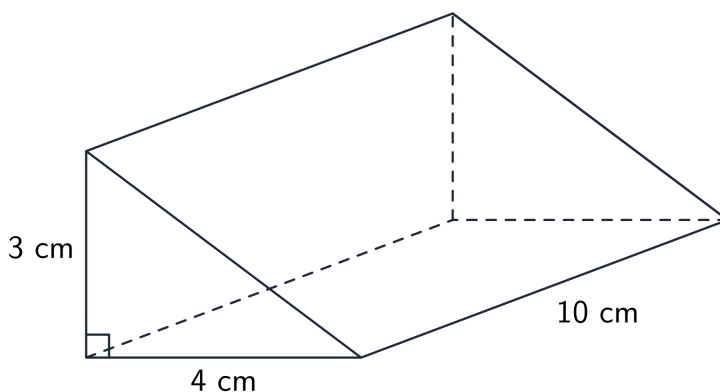
Plane geometry is about flat shapes, solid geometry about three-dimensional ones, and Euclidean geometry is the reasoning that connects them.

Angles on a straight line add to 180° , angles around a point to 360° , and the interior angles of a plane Euclidean triangle to 180° . In a simple **polygon** 多边形 of n sides the interior angles add to $(n - 2) \times 180^\circ$.

- **Congruent** 全等 shapes are identical in size and shape; **similar** 相似 shapes have the same shape with all lengths in one ratio.

- **Pythagoras' theorem** 勾股定理 holds in a right-angled triangle: $a^2 + b^2 = c^2$, where c is the **hypotenuse** 斜边.
- **Area** 面积 is measured in square units and **volume** 体积 in cubic units. A cylinder has volume $\pi r^2 h$; a sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.
- $\triangle$ Scaling is not linear. If every length of a solid is doubled, its area is multiplied by $2^2 = 4$ and its volume by $2^3 = 8$.

Worked example. a triangular prism has a constant cross-section



not to scale

Known: perpendicular triangle sides are 3 and 4 cm; prism length is 10 cm. Find the cross-sectional area, then multiply by the prism length.

$$A = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6 \text{ cm}^2$$

$$V = AL = 6(10) = 60 \text{ cm}^3$$

The sloping triangle side is not its perpendicular height. Volume is not the area of the triangular end.

Continue with practice sheet **1.4**. Solve before opening the solutions, and check both the method and the final units.

Trigonometry

Trigonometry connects the angles of a triangle to its sides.

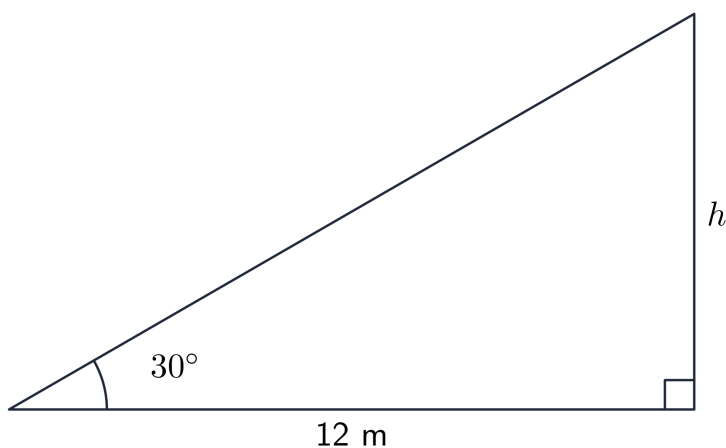
In a right-angled triangle, with θ one of the acute angles, the three ratios are $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$.

For a nondegenerate plane Euclidean triangle, the **sine rule** 正弦定理 and the **cosine rule** 余弦定理 apply:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad a^2 = b^2 + c^2 - 2bc \cos A$$

- An angle can be measured in **degrees** 度 or in **radians** 弧度, where π radians is 180° . Check which mode your calculator is in before every question.
- The **unit circle** 单位圆 extends the ratios beyond 90° and explains why sin and cos repeat every 360° : they are **periodic** 周期的.
- A **bearing** 方位角 is measured clockwise from north and always written with three figures, such as 075° .
- **Angles of elevation and depression** 仰角与俯角 are measured from the horizontal, upwards and downwards respectively. Add the observer or instrument height if the answer needs height above the ground. With side-side-angle data, the sine rule may allow two triangles: check both supplementary angles against the remaining angle sum.

Worked example. height from a horizontal distance and angle



not to scale

Known: horizontal distance $d = 12$ m and elevation $\theta = 30^\circ$. Height h is opposite and d is adjacent, so use tangent in degree mode.

$$\tan \theta = \frac{h}{d}$$

$$h = d \tan \theta = 12 \tan 30^\circ \approx 6.93 \text{ m}$$

This is the height above the observer's horizontal sight level, not automatically the height above the ground.

Continue with practice sheet **1.5**. Solve before opening the solutions, and check both the method and the final units.

Exponential and logarithmic functions

These two functions describe growth and decay, and they undo each other.

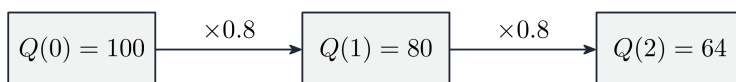
An **exponential function** 指数函数 has the form $y = a^x$ with $a > 0$ and $a \neq 1$: the variable is in the **exponent** 指数. Each unit step multiplies its value by a . It grows when $a > 1$ and decays when $0 < a < 1$. Its y-intercept is 1 and its values stay positive; the x-axis is a horizontal asymptote.

A **logarithm** 对数 answers the reverse question. $\log_a y = x$ means exactly $a^x = y$, so $\log_{10} 1000 = 3$. For real logarithms the argument must be positive, and the base must be positive and different from 1. The graph $y = \log_a x$ passes through $(1, 0)$ and has vertical asymptote $x = 0$.

$$\log(mn) = \log m + \log n, \quad \log\left(\frac{m}{n}\right) = \log m - \log n, \quad \log(m^k) = k \log m$$

- For positive arguments and the same valid base, the laws turn multiplication into addition, which is what makes a logarithm useful for solving an equation with the unknown in the exponent.
- **Natural logarithms** 自然对数 use the base $e \approx 2.718$ and are written $\ln$.
- **Compound interest** 复利 is exponential: $A = P(1 + r)^n$ after n periods.
- **Exponential decay** 指数衰减 has a base between 0 and 1, and can describe a retained proportion each period under stated assumptions. It does not guarantee that a real process follows that model indefinitely.

Worked example. repeated reductions multiply rather than subtract a fixed amount



equal time periods, changing absolute reductions

Known: an invented quantity starts at 100 units and decreases by 20% per whole period. The retained proportion is 0.8, so use a multiplicative model.

$$Q(n) = Q_0(1 - r)^n$$

$$Q(2) = 100(1 - 0.20)^2 = 64$$

The first decrease is 20 units and the second is 16. A fixed subtraction of 20 each period would be a different model.

Continue with practice sheet **1.6**. Solve before opening the solutions, and check both the method and the final units.

GAC010 Mathematics II: Probability, Statistics and Finance

GAC Mathematics

What this module is, and how it is marked

A cheaper headline rate need not produce the cheaper total, and a larger survey need not produce a more representative result. Mathematics II uses explicit rules, calculations and evidence limits to examine such questions.

GAC010 covers sequences, finance, probability and statistics. Your centre's current brief determines assessment tasks, weights, tools and deadlines. A **test** 测验, **projects** 项目, an **examination** 考试 and **coursework** 平时作业 may use different instructions; these original practice sheets do not establish their official pattern or university credit.

Show the known data, why the formula applies, the symbolic equation and the substitution. State the unit and interpret the result. A statistical summary describes a defined set of observations or a stated model; it does not automatically establish a cause or a whole-population finding.

All financial amounts, rates, fees and pay rules in the examples are invented teaching conditions. Use their stated timing and rounding assumptions rather than assuming an actual tax or lending rule.

Sequences and series

A **sequence** 数列 is an ordered list of numbers; a **series** 级数 is their sum.

An **arithmetic sequence** 等差数列 adds a fixed **common difference** 公差 d each step. A **geometric sequence** 等比数列 multiplies by a fixed **common ratio** 公比 r .

$$u_n = a + (n - 1)d \quad \text{and} \quad u_n = ar^{n-1}$$

- The sums are $S_n = \frac{n}{2}(2a + (n - 1)d)$ and $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r \neq 1$.
- A geometric series with $|r| < 1$ has a **sum to infinity** 无穷和 $S_\infty = \frac{a}{1 - r}$.
- Keep a term separate from a total. Some financial models compound geometrically, while others use simple interest or cash flows. Match the model to the specified rule.

Worked example. row counts differ by a constant amount

row 1: 4 • • • •

row 2: 7 • • • • • • •

row 3: 10 • • • • • • • • • •

Known: row counts are 4, 7 and 10, continuing with the same difference. Thus $a = 4$ and $d = 3$. For the fifth row:

$$u_n = a + (n - 1)d$$

$$u_5 = 4 + (5 - 1)(3) = 16$$

For the first five rows together, use the arithmetic sum, not the fifth term.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_5 = \frac{5}{2}[2(4) + (5 - 1)(3)] = 50$$

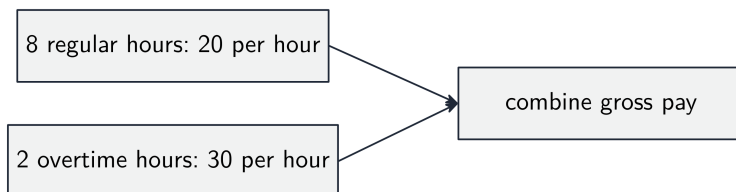
Practice sheet **2.1** gives a progressively harder set with worked solutions.

Finance: earning money and purchasing goods

This unit is the arithmetic of an ordinary life, in English.

- **Gross pay** 税前工资 is what you earn; **net pay** 税后工资 is what arrives after **deductions** 扣除项 such as tax and insurance.
- **Simple interest** 单利 is $I = Prt$ — interest on the original amount only.
- **Compound interest** is $A = P(1 + r)^n$ — interest on the interest, which is why it is geometric.
- **Depreciation** 折旧 is a reduction in recorded value. A fixed-percentage model uses $V_n = V_0(1 - r)^n$; other depreciation conventions can use different rules.
- A **discount** 折扣 of 20% multiplies by 0.8. Two successive discounts of 20% and 10% multiply by $0.8 \times 0.9 = 0.72$, which is a 28% reduction, not 30%.

Worked example. calculate regular and overtime hours separately



Known: 8 hours at 20 units per hour and 2 overtime hours at 1.5 times that rate. Separate the two categories before adding pay.

$$G = h_r p + h_o k p$$

$$G = 8(20) + 2(1.5)(20) = 220 \text{ units}$$

If the invented fixed deduction is 15 units, take it from gross pay once.

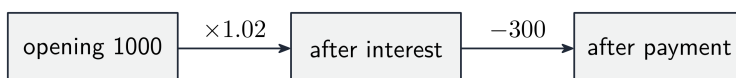
$$N = G - D = 220 - 15 = 205 \text{ units}$$

Practice sheet **2.2** gives a progressively harder set with worked solutions.

Finance: investing money and loans

- Under a stated constant discount rate and timing assumption, the **present value** 现值 of a future amount is: $PV = \frac{FV}{(1+r)^n}$.
- An **annuity** 年金 has specified payments at regular intervals. A **loan repayment** 贷款还款 schedule may instead have changing amounts or an adjusted final payment; specify whether interest comes before or after payment.
- The label **APR** 年利率 is an annual rate measure whose calculation and included charges depend on the stated convention. This sheet uses explicitly defined period rates, not an unstated APR convention.
- $\triangle$ A nominal 12% teaching rate with monthly rate 1% has effective annual growth $(1.01)^{12} - 1 \approx 12.68\%$. A different compounding frequency or fee schedule needs a different calculation.

Worked example. interest then payment, in that order



Known: opening balance 1000 units, period rate 2%, end-of-period payment 300 units. Interest comes before payment under this stated convention.

$$B_1 = B_0(1 + r) - M$$

$$B_1 = 1000(1.02) - 300 = 720$$

The next period's interest uses 720, not the original principal.

$$B_2 = B_1(1 + r) - M = 720(1.02) - 300 = 434.40$$

Practice sheet **2.3** gives a progressively harder set with worked solutions.

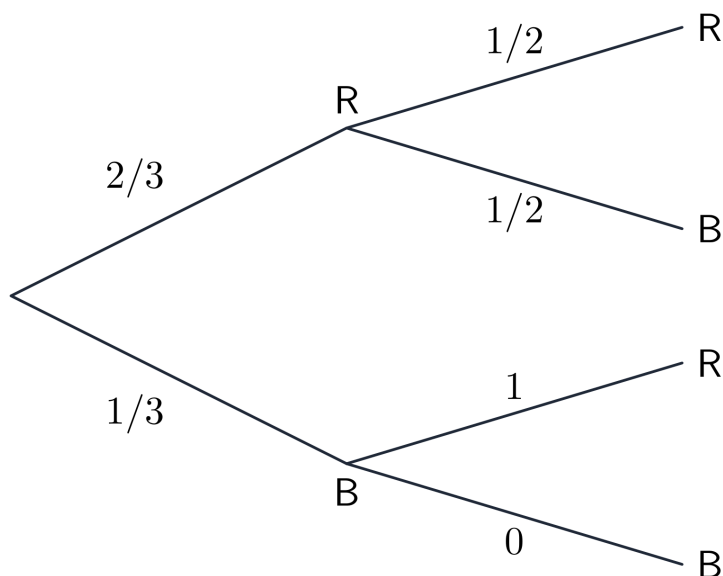
Probability: key concepts

Probability 概率 measures how likely an event is, on a scale from 0 to 1.

- For equally likely outcomes, $P(A) = \frac{\text{favourable outcomes}}{\text{total outcomes}}$.
- Events are **mutually exclusive** 互斥 when they cannot both happen: $P(A \text{ or } B) = P(A) + P(B)$.
- Events are **independent** 独立 when one does not affect the other: $P(A \text{ and } B) = P(A) \times P(B)$.
- The **complement** 对立事件 rule, $P(\text{not } A) = 1 - P(A)$, is often the fastest route to an answer containing “at least one”.
- **Counting techniques** 计数方法 supply the denominator: a **permutation** 排列 counts ordered selections, a **combination** 组合 counts unordered ones.

Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$ for $P(B) > 0$. A union subtracts its overlap: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Association in a supplied table does not by itself establish a cause.

Worked example. a second draw depends on the first



Known: two red and one blue counter, two random draws without replacement. To get two reds, follow the red-red path and multiply its conditional probabilities.

$$P(RR) = P(R_1)P(R_2 | R_1)$$

$$P(RR) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

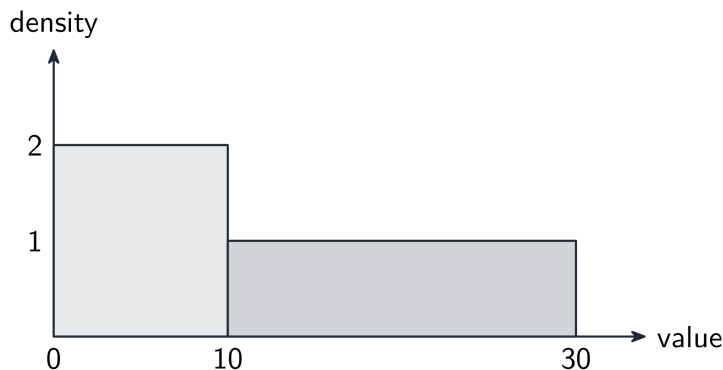
A first blue leaves only red counters; the blue-blue path has probability zero. The draws are not independent.

Practice sheet **2.4** gives a progressively harder set with worked solutions.

Statistics: collecting and displaying data

- A **population** 总体 is everyone you want to describe; a **sample** 样本 is who you actually measure.
- Generalising to a population needs a suitably **representative** 有代表性的 sample and an honest account of selection and nonresponse. A class survey can describe that class without establishing a school-wide percentage.
- **Qualitative data** 定性数据 are categories; **quantitative data** 定量数据 measure amounts, either **discrete** 离散 (counted) or **continuous** 连续 (measured).
- **Bias** 偏差 enters through who you ask, who answers, and how the question is worded, and arithmetic alone does not repair missing or systematically selected observations.

Worked example. equal bar heights can conceal unequal frequencies



Known: class 0 to 10 has frequency 20; class 10 to 30 has frequency 20. Widths differ, so calculate frequency density.

$$D = \frac{f}{w}$$

$$D_1 = \frac{20}{10} = 2$$

$$D_2 = \frac{20}{20} = 1$$

The second bar is half as high but twice as wide. Both areas represent 20 observations.

Practice sheet **2.5** gives a progressively harder set with worked solutions.

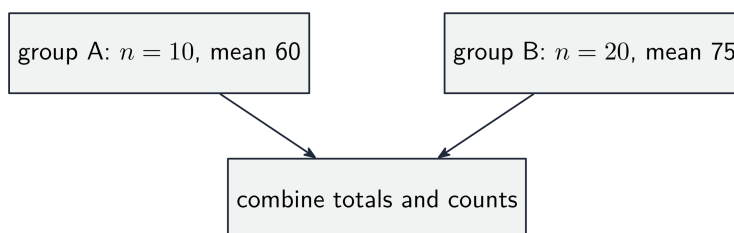
For unequal-width histogram classes, height is frequency divided by class width so that bar area represents frequency. Keep categories in separated bars and numeric continuous intervals in a histogram.

Statistics: measures of central location

- The **mean** 平均数 is the total divided by the count. It uses every value, and every outlier.
- The **median** 中位数 is the middle value in order. Moving an extreme value without changing the ordered middle may leave it unchanged.
- The **mode** 众数 is the most frequent value, and it can identify the most frequent category without imposing a numerical scale. A distribution may have more than one mode or no unique mode.

- $\triangle$ Choose the summary for the question. A high extreme can pull the mean above the median. The mean still measures total per observation; the median locates the ordered middle. Neither says that most observations equal it.

Worked example. group size determines a combined mean



Known: group A has 10 scores with mean 60; group B has 20 with mean 75. Recover each group's total before combining.

$$\bar{x} = \frac{n_A \bar{x}_A + n_B \bar{x}_B}{n_A + n_B}$$

$$\bar{x} = \frac{10(60) + 20(75)}{10 + 20} = 70$$

The unweighted mean of 60 and 75 would be 67.5, which ignores the unequal counts.

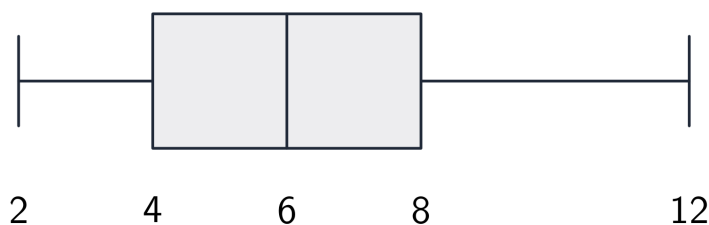
Practice sheet **2.6** gives a progressively harder set with worked solutions.

Statistics: measures of variability

- The **range** 全距 is the largest minus the smallest — the full span, so an extreme can change it substantially.
- The **interquartile range (IQR)** 四分位距 is the difference $Q_3 - Q_1$ spanning the central half by the stated quartile convention. A **box plot** 箱线图 shows quartiles and a median, with whisker rules that must be specified.
- The **standard deviation** 标准差 summarises spread using squared deviations. For a complete population, use: $\sigma = \sqrt{\sum(x - \mu)^2/n}$. A sample estimate uses $s = \sqrt{\sum(x - \bar{x})^2/(n - 1)}$ under that stated convention. Variance has squared units; standard deviation has the original units.
- A larger standard deviation means a wider spread. Two data sets can share a mean and describe completely different situations.

Worked example. read a five-number summary rather than every observation

whiskers: minimum to maximum



Known five-number summary: minimum 2, $Q_1 = 4$, median 6, $Q_3 = 8$, maximum 12. The overall span and middle-half span are different.

$$R = x_{max} - x_{min} = 12 - 2 = 10$$

$$IQR = Q_3 - Q_1 = 8 - 4 = 4$$

The box does not identify every original value, the sample size or why the spread arose.

Practice sheet **2.7** gives a progressively harder set with worked solutions.

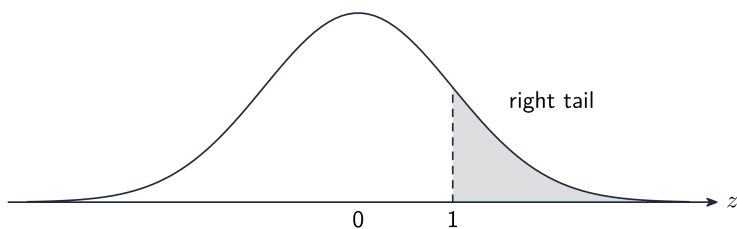
For sheet 2.7, quartiles are medians of ordered half-lists, excluding the overall median for an odd count. Whiskers run to the minimum and maximum. Do not mix that convention with software interpolated quartiles or outlier-fence whiskers.

Distributions: the normal distribution

- The **normal distribution** 正态分布 is a continuous symmetric bell-shaped model specified by its mean and positive standard deviation. Those two summaries alone do not establish that observed data follow this model.
- Under a normal model, the approximate empirical rule is: about 68% of values lie within one standard deviation of the mean, 95% within two, and 99.7% within three.
- A **z-score** 标准分 says how many standard deviations a value sits from the mean: $z = \frac{x - \mu}{\sigma}$, which is what makes two different scales comparable.

For a normal model, use cumulative area $\Phi(z) = P(Z \leq z)$ and symmetry $\Phi(-z) = 1 - \Phi(z)$. An interval subtracts its two cumulative bounds. Expected count is sample size times probability and need not be an integer; an observed count is an integer.

Worked example. a threshold is an area beyond a standardised value



Known: an invented measurement model has $\mu = 50$ and $\sigma = 10$. For values above 60, standardise the threshold.

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 50}{10} = 1$$

Use the complement of the supplied cumulative area.

$$P(X > 60) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

This is a model probability, not a guarantee that exactly 15.87% of a finite sample exceeds 60.

Practice sheet **2.8** gives a progressively harder set with worked solutions.

GAC016 Mathematics III: Calculus & Advanced Applications

GAC Mathematics

What this module is, and how it is marked

An average speed does not tell you every instantaneous speed, and a signed displacement does not always equal total distance. Calculus makes those distinctions precise.

GAC016 develops **differentiation** 微分, **integration** 积分 and their applications. Your centre's current brief determines assessment tasks and grading requirements; these original practice sheets do not establish a university credit decision or an official examination pattern.

State the independent variable, domain and units. The notation dy/dx is a derivative with respect to x ; a dot over y normally denotes a derivative with respect to time. Those notations describe the same operation only when their independent variables agree.

For applications, describe what a rate or accumulated value means in the stated model. A negative volume-change rate of 3 cubic centimetres per second means volume is falling at that rate, rather than that the volume itself is negative.

Differentiation

A **derivative** 导数 is a rate of change: how fast y changes as x changes. Geometrically it is the **gradient of the tangent** 切线斜率 at a point.

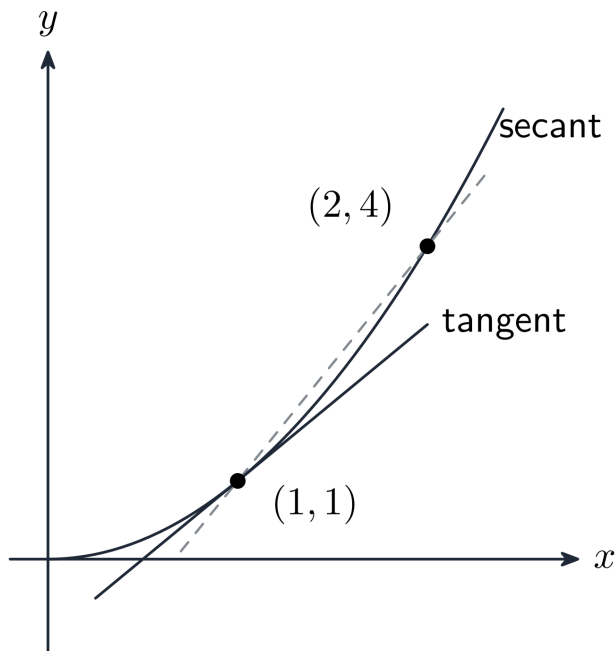
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

That limit is the definition. **Differentiation from first principles** 从定义求导 uses the difference quotient for nonzero h and then takes its limit. Unequal one-sided limits, as for $|x|$ at zero, prevent a derivative even when the function is continuous.

- The **power rule** 幂法则, on a domain where the power is differentiable: if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.
- The **product rule** 乘积法则: $(uv)' = u'v + uv'$.
- The **quotient rule** 商法则, where $v \neq 0$: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.
- The **chain rule** 链式法则: if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.

- **Stationary points** 驻点 occur where $f'(x) = 0$. The **second derivative** 二阶导数 then classifies a stationary point when it is nonzero: negative gives a local **maximum** 极大值 and positive a local **minimum** 极小值. A zero second derivative is inconclusive; use derivative signs or other evidence. A stationary inflection can be neither an extremum.

Worked example. a tangent slope differs from a secant slope



Known: $f(x) = x^2$. The secant between x equal to 1 and 2 has slope 3, but the derivative at x equal to 1 is 2.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

$$m_{tan} = f'(1) = 2(1) = 2$$

The average slope over a finite interval need not equal the instantaneous slope at either endpoint.

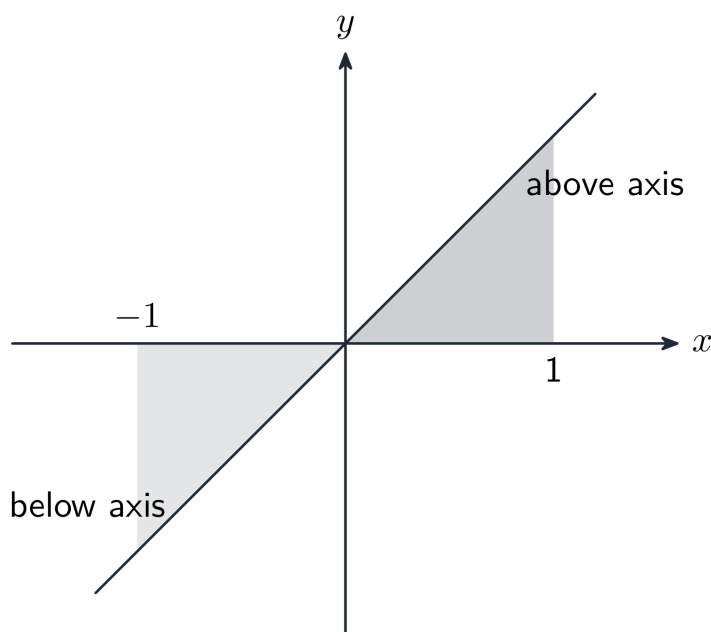
Use original practice sheet **3.1** to test the method, domain and interpretation against its solutions.

Integration

Indefinite integration finds antiderivatives. Definite integration measures signed accumulation, which can differ from geometric area. For a continuous integrand, the **fundamental theorem of calculus** 微积分基本定理 connects a definite integral to the difference of antiderivative values at its limits.

- An **indefinite integral** 不定积分 has no limits and needs the **constant of integration** 积分常数: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** 定积分 has limits and gives a number: $\int_a^b f(x) dx = F(b) - F(a)$.
- **Integration by substitution** 换元积分法 reverses the chain rule.
- $\triangle$ Forgetting $+c$ on an indefinite integral is the single most frequent lost mark in the module, and it is lost on questions you have otherwise answered correctly.

Worked example. negative and positive areas can cancel



Known: $f(x) = x$ from negative 1 to 1. Its definite integral is zero because the negative and positive contributions cancel.

$$I = \int_{-1}^1 x dx = \left[\frac{x^2}{2} \right]_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$$

Geometric area instead adds the magnitudes of the two triangular regions.

$$A = - \int_{-1}^0 x \, dx + \int_0^1 x \, dx = \frac{1}{2} + \frac{1}{2} = 1$$

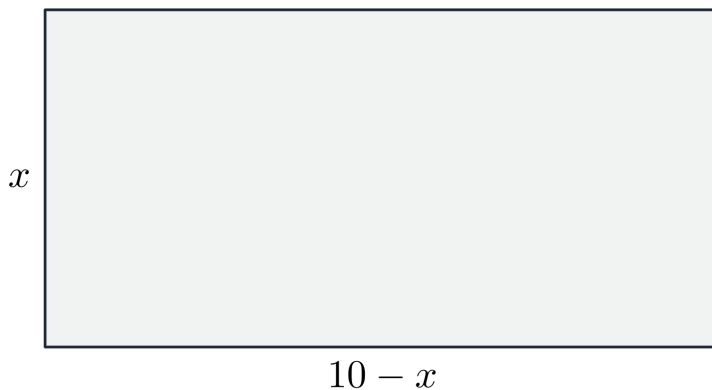
Use original practice sheet **3.2** to test the method, domain and interpretation against its solutions.

Advanced applications

- **Optimisation** 最优化 finds the largest or smallest value of a quantity: write the quantity as a function of one variable on a stated feasible domain, differentiate, find candidates and justify the required optimum. Check included boundaries and integer constraints where relevant.
- **Rates of change** 变化率 chain together: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$.
- **Area between two curves** 两曲线间面积 is $\int_a^b (f(x) - g(x)) \, dx$ where f is the upper curve on that whole interval. Split at crossings if their order changes.
- **Marginal cost and revenue** 边际成本与边际收益 are derivatives of stated continuous cost and revenue models. They are instantaneous rates, not automatically exact discrete-unit changes.

Worked example. a fixed perimeter leaves one area variable

perimeter 20 units



not to scale

Known: a rectangle has perimeter 20, so length plus width is 10. Let width be x ; length is $10 - x$ and $0 < x < 10$.

$$A(x) = x(10 - x)$$

$$A'(x) = 10 - 2x$$

The stationary candidate is $x = 5$. Since $A''(x) = -2 < 0$ and the area approaches zero at both domain ends, this candidate is the global maximum. It is a 5 by 5 square with area 25.

Use original practice sheet **3.3** to test the method, domain and interpretation against its solutions.

GAC024 Discrete Mathematics

GAC Mathematics

What this module is, and how it is marked

GAC024 is discrete mathematics: the mathematics of things you can count and of the logic computers run on. Five units cover sets, counting systems, binary, algorithms, and graphs.

It pairs naturally with the computing modules, and universities that credit GAC016 often also recognise this one for a computer-science foundation course. Assessment follows the usual pattern: a test, an examination, and coursework.

- $\triangle$ Discrete mathematics has few formulas and much **reasoning**. A proof or a traced algorithm carries the marks, and an unexplained answer earns almost nothing.

Sets, relations and functions

- A **set** 集合 is a collection of distinct objects. Order and repetition do not matter.
- **Union** 并集 $A \cup B$ is everything in either; **intersection** 交集 $A \cap B$ is what is in both; the **set complement** 补集 is everything outside.
- A **subset** 子集 has all its elements inside another set.
- A **relation** 关系 pairs elements of two sets. A **function** 函数 is a relation where each input has exactly one output.
- A **Venn diagram** 韦恩图 turns a set problem into a picture, and drawing one is usually faster than reasoning about it in words.

Worked example. In a class of 30, 18 study French and 15 study German; 7 study both. How many study neither?

$$|F \cup G| = 18 + 15 - 7 = 26 \quad \Rightarrow \quad 30 - 26 = 4$$

Subtracting the overlap once is the **inclusion-exclusion principle** 容斥原理. Adding 18 and 15 without it counts the seven twice, which is the standard error.

Counting systems

- A **number base** 进制 says how many digits it uses. **Decimal** 十进制 uses ten, **binary** 二进制 two, **hexadecimal** 十六进制 sixteen.

- Every digit's value is its **place value** 位值: in binary the places are 1, 2, 4, 8, 16 and so on.
- Hexadecimal is shorthand for binary: one hex digit is exactly four bits, which is why memory addresses are written in it.

Worked example. Convert 1101 in binary to decimal.

$$8 + 4 + 0 + 1 = 13$$

Write the place values above the digits before you add. Doing it in your head is where the off-by-one errors come from.

Binary applications

- **Binary arithmetic** 二进制运算 adds like decimal, carrying at 2 instead of at 10.
- A **bit** 位 is one binary digit; a **byte** 字节 is eight.
- **Boolean algebra** 布尔代数 works on true and false with AND, OR and NOT.
- A **truth table** 真值表 lists every input combination and its output, and it is the complete proof that two logic expressions are equivalent.
- **Logic gates** 逻辑门 are the physical form of those operations, and a **logic circuit** 逻辑电路 is what a processor is made of.

Algorithms

- An **algorithm** 算法 is a finite sequence of unambiguous steps that terminates.
- A **flowchart** 流程图 draws it: a decision is a diamond, a process a rectangle.
- **Pseudocode** 伪代码 writes it in structured English, which is what an exam usually asks for.
- **Tracing** 追踪 an algorithm — a table with one column per variable and one row per step — is the marked technique, and it is how you find a bug without running anything.
- **Efficiency** 效率 matters: a linear search checks every item, a binary search halves the list each time, and on a million items that is the difference between a million steps and twenty.

Worked example. Trace a binary search for 7 in [1, 3, 5, 7, 9, 11].

Middle is 5, which is less than 7, so search the right half. Middle of [7, 9, 11] is 9, which is greater, so search the left. Middle of [7] is 7. Found, in three steps rather than four.

The table of steps is the answer. The word “found” is not.

Graphs and networks

- A **graph** 图 is a set of **vertices** 顶点 joined by **edges** 边. It models anything with connections: roads, friendships, dependencies.
- The **degree** 度 of a vertex is how many edges meet it.
- A **tree** 树 is a connected graph with no cycles, and it is the shape of a file system, an organisation chart and an HTML document.
- A **shortest path** 最短路径 problem asks for the cheapest route between two vertices, and it is what a navigation app solves every time you use it.