GAC016 Mathematics III: Calculus & Advanced Applications

GAC Mathematics

What this module is, and how it is marked

GAC016 is calculus, and it is the GAC module universities most often award credit for. Several publish a first-year calculus equivalency against it, which means the standard expected here is a university standard, not a school one.

Three units only — **differentiation** 微分, **integration** 积分, and applications — so each is covered in more depth than a Level I unit. Assessment follows the usual pattern: a test, an examination over all three units, and coursework.

- △ Notation is marked. $\frac{dy}{dx}$, $f'(x)$ and $\dot{y}$ mean the same thing, but a question that uses one expects it back.
- Every applied answer needs its **units** and its **interpretation**. "-3" is not an answer; "the volume is falling at 3 cm^3 per second" is.

Differentiation

A **derivative** 导数 is a rate of change: how fast y changes as x changes. Geometrically it is the **gradient of the tangent** 切线斜率 at a point.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

That limit is the definition, and **differentiation from first principles** 从定义求导 is examinable even though you will use the rules afterwards.

- The **power rule** 幂法则: if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.
- The **product rule** 乘积法则: $(uv)' = u'v + uv'$.
- The **quotient rule** 商法则: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.
- The **chain rule** 链式法则: if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.
- **Stationary points** 驻点 occur where $f'(x) = 0$. The **second derivative** 二阶导数 then classifies them: negative is a **maximum** 极大值, positive a **minimum** 极小值.

Worked example. Find and classify the stationary points of $y = x^3 - 3x$.

$$\frac{dy}{dx} = 3x^2 - 3 = 0 \;\Rightarrow\; x = \pm 1$$

$$\frac{d^2y}{dx^2} = 6x$$

At $x = 1$ the second derivative is $6 > 0$, so it is a minimum at $(1, -2)$. At $x = -1$ it is $-6 < 0$, so a maximum at $(-1, 2)$. Classifying with the second derivative is the marked step; saying "one is a maximum" without it earns nothing.

Integration

Integration is differentiation run backwards, and also the area under a curve. Those two facts are the same fact, which is what the **fundamental theorem of calculus** 微积分基本定理 says.

- An **indefinite integral** 不定积分 has no limits and needs the **constant of integration** 积分常数: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** 定积分 has limits and gives a number: $\int_a^b f(x)\,dx = F(b) - F(a)$.
- **Integration by substitution** 换元积分法 reverses the chain rule.
- △ Forgetting $+c$ on an indefinite integral is the single most frequent lost mark in the module, and it is lost on questions you have otherwise answered correctly.

Worked example. Find the area between $y = x^2$ and the x-axis from $x = 0$ to $x = 3$.

$$\int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = 9 - 0 = 9$$

The square brackets with limits are part of the method, not decoration. An answer of 9 with no bracket line shows nothing about how it was obtained.

Advanced applications

- **Optimisation** 最优化 finds the largest or smallest value of a quantity: write the quantity as a function of one variable, differentiate, set to zero, and classify.
- **Rates of change** 变化率 chain together: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$.

- **Area between two curves** 两曲线间面积 is $\int_a^b (f(x) - g(x))\,dx$ where f is the upper curve.
- **Marginal cost and revenue** 边际成本与边际收益 are derivatives, which is why economics departments credit this module.

Worked example. A cylindrical can must hold 500 cm^3. What radius minimises the metal used?

Surface area $A = 2\pi r^2 + 2\pi rh$, and the volume constraint gives $h = \dfrac{500}{\pi r^2}$.

$$A = 2\pi r^2 + \frac{1000}{r}, \qquad \frac{dA}{dr} = 4\pi r - \frac{1000}{r^2} = 0$$

So $r^3 = \dfrac{250}{\pi}$ and $r = 4.30$ cm. The constraint is what turns two variables into one, and writing that substitution line is where most of the method marks sit.