

4.5 Analyse simple graphs, trees and weighted routes

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS vertices 顶点 • edges 边 • tree 树

Objective

Analyse simple graphs, trees and weighted routes. Original GAC004-aligned practice, not an official assessment. Show the method as well as the result. All applied scenarios are invented classroom problems.

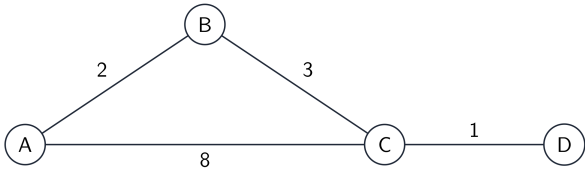
1 Key knowledge and worked examples

■ **Key knowledge: state whether connections are directed, weighted or undirected**

A graph has **vertices** 顶点 connected by **edges** 边. This sheet uses finite simple undirected graphs unless a question says otherwise: no loops or repeated edges. Degree counts incident edges; the sum of degrees is twice the edge count. A path connects vertices, and a cycle returns to its start without repeating other vertices.

A **tree** 树 is connected and has no cycles; a tree with n vertices has n minus 1 edges. That edge count alone does not prove a graph is a tree without connectedness. In a weighted network, a shortest path minimises total path weight, whereas a minimum spanning tree connects all vertices with minimum total edge weight and no cycle. These are different optimisation tasks.

■ **Worked example: compare total route weight, not the number of edges**



Known edge weights are $AB = 2$, $BC = 3$, $AC = 8$ and $CD = 1$. The path A-C-D has weight 9, while A-B-C-D has weight 6. Therefore the three-edge path is shorter by weight despite having more edges. The minimum spanning tree for this small network uses AB, BC and CD with total 6; the agreement of totals here does not make the tasks identical.

2 Practice

2.1 A simple undirected graph has edges AB, AC and BC. Find the degree of A and the total edge count. [2]

2.2 A tree has 9 vertices. Find its edge count and state the property excluding a triangle cycle. [2]

2.3 A simple undirected graph has degrees 3, 2, 2, 1. Find its edge count and explain why degree sum 9 would be impossible. [3]

2.4 A graph has edges AB, BC, CA and DE, on exactly five vertices A to E. It has n minus 1 edges. Decide whether it is a tree and justify your decision using two graph properties. [3]

3 Exam-style questions

3.1 A weighted undirected network has edges $AB = 2$, $AC = 6$, $BC = 1$, $BD = 7$, $CD = 3$. (a) Find a shortest path from A to D and its weight. (b) Find a minimum spanning tree and its total weight. (c) Explain the different requirements of these tasks. [6]

3.2 A weighted undirected network on A, B, C, D and E has edges $AB = 1$, $AC = 4$, $BC = 2$, $BD = 5$, $CD = 1$, $CE = 6$, $DE = 3$. (a) Find a minimum spanning tree and its total weight, showing how cycles are avoided. (b) Find a shortest path from A to E and its weight. (c) If DE closes, find a new shortest A-to-E route using the remaining edges. Explain why that new route must be checked rather than retaining the old total. [8]

Solutions

2.1

A is incident to AB and AC, so degree 2. Total edge count 3.

2.2

A tree has $n - 1 = 9 - 1 = 8$ edges. It is acyclic, so no triangle cycle is allowed.

2.3

Each edge contributes two to total degree.

$$|E| = \frac{\sum d(v)}{2} = \frac{3 + 2 + 2 + 1}{2} = 4$$

Four edges. A sum of 9 is odd and cannot equal twice an integer edge count.

2.4

It is not a tree. The ABC triangle is a cycle. The DE component is disconnected from ABC. The four-edge count alone is insufficient.

3.1

(a) A-B-C-D has weight $2 + 1 + 3 = 6$. Correct path earns B1 and weight B1. The other simple A-to-D alternatives are A-B-D with 9, A-C-D with 9, and A-C-B-D with 14, so 6 is shortest.

(b) Choose BC, AB and CD to connect all vertices without a cycle. Total weight 6.

(c) The shortest-path task connects only the specified endpoints with minimum route weight. The spanning-tree task connects all vertices with no cycles and minimum total included edge weight.

3.2

(a) Select AB and CD, both weight 1. Add BC with weight 2, joining those components without a cycle. Add DE with weight 3 to connect E. Total $W = 1 + 1 + 2 + 3 = 7$, with all five vertices connected and four edges.

(b) Shortest path A-B-C-D-E. Weight $W_p = 1 + 2 + 1 + 3 = 7$. Nonnegative edge weights and comparison with alternative approaches to C/D establish no shorter route.

(c) E is now accessible only through CE, so the shortest remaining route is A-B-C-E with weight $1 + 2 + 6 = 9$. Correct route and weight. The closed edge cannot be used; the old path is infeasible and its total is not a valid remaining-network answer.