

3.3 Apply calculus with physical and feasible-domain checks

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS optimisation 最优化 • rates of change 变化率 • marginal cost 边际成本

Objective

Apply calculus with physical and feasible-domain checks. Original GAC004-aligned practice, not an official assessment. Show the method as well as the result. All applied scenarios are invented classroom problems.

1 Key knowledge and worked examples

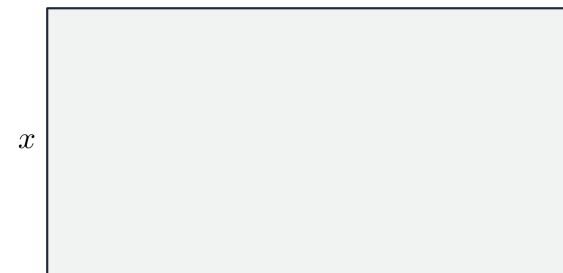
■ **Key knowledge:** a stationary candidate is not automatically the required optimum

Optimisation 最优化 starts with a quantity and its feasible domain. Use a constraint to express it in one variable, find stationary candidates, compare boundaries where applicable, and justify the required maximum or minimum. A continuous optimum may differ from the best feasible integer choice.

Related **rates of change** 变化率 use the chain rule: $dV/dt = (dV/dr)(dr/dt)$ when V depends on r and r on time. **Marginal cost** 边际成本 is the derivative of a continuous cost model, an instantaneous rate rather than automatically an exact discrete-unit increase. Integrating a rate gives an accumulated change; include an initial value when finding a final amount. All scenarios here are invented mathematical models.

■ **Worked example:** a fixed perimeter leaves one area variable

perimeter 20 units



$10 - x$

not to scale

Known: a rectangle has perimeter 20, so length plus width is 10. Let width be x ; length is $10 - x$ and $0 < x < 10$.

$$A(x) = x(10 - x)$$

$$A'(x) = 10 - 2x$$

The stationary candidate is $x = 5$. Since $A''(x) = -2 < 0$ and the area approaches zero at both domain ends, this candidate is the global maximum. It is a 5 by 5 square with area 25.

2 Practice

2.1 A continuous model has cost $C(q) = 100 + 3q + 0.2q^2$. Find its marginal cost at q equal to 10. [2]

2.2 A constant flow adds water at $r(t) = 6$ litres per minute for 5 minutes. Find the increase in volume. [2]

2.3 A circle radius grows at $dr/dt = 0.5$ cm per second. Find its area growth rate when r equals 4 cm. [3]

2.4 For $C(q) = 100 + 3q + 0.2q^2$, compare $C(11) - C(10)$ with $C'(10)$. Explain the difference. [3]

3 Exam-style questions

3.1 A rectangle has perimeter 40 m. (a) Express area in terms of width x, state its feasible domain, and find the width maximising area. (b) Justify the maximum and find the maximum area. [6]

3.2 A continuous teaching profit model is $P(q) = -q^2 + 11q - 10$ for $0 \leq q \leq 10$. (a) Find and justify its continuous maximum. (b) If q must instead be a whole number, find all quantities giving the largest feasible profit. (c) Find the break-even quantities and the whole-number quantities giving strictly positive profit. [8]

Solutions

2.1

Differentiate the cost model.

$$C'(q) = 3 + 0.4q$$

$$C'(10) = 3 + 0.4(10) = 7$$

Marginal cost 7 cost units per quantity unit.

2.2

Integrate the rate over the stated time.

$$\Delta V = \int_0^5 r(t)dt = \int_0^5 6 dt = 30 \text{ litres}$$

Increase 30 litres.

2.3

Use $A = \pi r^2$ with the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

Correct related-rate relation.

$$\frac{dA}{dt} = 2\pi(4)(0.5) = 4\pi \frac{\text{cm}^2}{\text{s}}$$

Correct rate.

2.4

Compute the actual one-unit change.

$$\Delta C = C(11) - C(10) = 157.2 - 150 = 7.2$$

Correct finite increase 7.2. The derivative at 10 is 7, an instantaneous rate; the derivative changes over the next unit, so it need not equal the finite increase.

3.1

The length is $20 - x$, with $0 < x < 20$.

$$A(x) = x(20 - x)$$

Correct area model. Differentiate and set to zero.

$$A'(x) = 20 - 2x = 0 \implies x = 10$$

Width 10 m. The second derivative is negative 2 and area approaches zero at the end-points, so this is the global maximum on the feasible domain.

$$A_{max} = 10(20 - 10) = 100 \text{ m}^2$$

Maximum area.

3.2

(a) Differentiate.

$$P'(q) = -2q + 11 = 0 \implies q = 5.5$$

Stationary quantity. Since $P''(q) = -2 < 0$ and both boundary profits are smaller, it is the global maximum.

$$P(5.5) = -(5.5)^2 + 11(5.5) - 10 = 20.25$$

Continuous maximum profit.

(b) Check the adjacent integers 5 and 6.

$$P(5) = -25 + 55 - 10 = 20$$

$$P(6) = -36 + 66 - 10 = 20$$

Both quantities maximise feasible integer profit.

(c) Factor the zero-profit equation: $P(q) = -(q - 1)(q - 10) = 0$, giving break-even quantities 1 and 10. Strictly positive profit is $1 < q < 10$, hence whole numbers 2 through 9.