

Exercise walk-through

3.3

Advanced Applications

GAC Mathematics

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The method

Method — Key knowledge: a stationary candidate is not automatically the required optimum

Optimisation 最优化 starts with a quantity and its feasible domain. Use a constraint to express it in one variable, find stationary candidates, compare boundaries where applicable, and justify the required maximum or minimum. A continuous optimum may differ from the best feasible integer choice.

Related **rates of change 变化率** use the chain rule: $dV/dt = (dV/dr)(dr/dt)$ when V depends on r and r on time. **Marginal cost 边际成本** is the derivative of a continuous cost model, an instantaneous rate rather than automatically an exact discrete-unit increase. Integrating a rate gives an accumulated change; include an initial value when finding a final amount. All scenarios here are invented mathematical models.

Method — Worked example: a fixed perimeter leaves one area variable

Known: a rectangle has perimeter 20, so length plus width is 10. Let width be x ; length is $10 - x$ and $0 < x < 10$.

$$A(x) = x(10 - x)$$

$$A'(x) = 10 - 2x$$

The stationary candidate is $x = 5$. Since $A''(x) = -2 < 0$ and the area approaches zero at both domain ends, this candidate is the global maximum. It is a 5 by 5 square with area 25.

Exercise 2.1

2.1 A continuous model has cost $C(q) = 100 + 3q + 0.2q^2$. Find its marginal cost at q equal to 10. [2]

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Differentiate the cost model.

$$C'(q) = 3 + 0.4q$$

$$C'(10) = 3 + 0.4(10) = 7$$

Marginal cost 7 cost units per quantity unit.

Exercise 2.2

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Integrate the rate over the stated time.

$$\Delta V = \int_0^5 r(t) dt = \int_0^5 6 dt = 30 \text{ litres}$$

Increase 30 litres.

Exercise 2.3

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Use $A = \pi r^2$ with the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

Correct related-rate relation.

$$\frac{dA}{dt} = 2\pi(4)(0.5) = 4\pi \frac{\text{cm}^2}{\text{s}}$$

Correct rate.

Exercise 2.4

2.4 For $C(q) = 100 + 3q + 0.2q^2$, compare $C(11) - C(10)$ with $C'(10)$. Explain the difference. [3]

3 Exam-style questions

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Compute the actual one-unit change.

$$\Delta C = C(11) - C(10) = 157.2 - 150 = 7.2$$

Correct finite increase 7.2. The derivative at 10 is 7, an instantaneous rate; the derivative changes over the next unit, so it need not equal the finite increase.

Exercise 3.1

3.1 A rectangle has perimeter 40 m. (a) Express area in terms of width x , state its feasible domain, and find the width maximising area. (b) Justify the maximum and find the maximum area. [6]

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3.1

The length is $20 - x$, with $0 < x < 20$.

$$A(x) = x(20 - x)$$

$$A'(x) = 20 - 2x = 0 \implies x = 10$$

Width 10 m. The second derivative is negative 2 and area approaches zero at the end-points, so this is the global maximum on the feasible domain.

$$A_{max} = 10(20 - 10) = 100 \text{ m}^2$$

Maximum area.

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(a) Differentiate.

$$P'(q) = -2q + 11 = 0 \implies q = 5.5$$

Stationary quantity. Since $P''(q) = -2 < 0$ and both boundary profits are smaller, it is the global maximum.

$$P(5.5) = -(5.5)^2 + 11(5.5) - 10 = 20.25$$

Continuous maximum profit.

Full solution 3.2 (1/2)

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(a) Differentiate.

$$P'(q) = -2q + 11 = 0 \implies q = 5.5$$

Stationary quantity. Since $P''(q) = -2 < 0$ and both boundary profits are smaller, it is the global maximum.

$$P(5.5) = -(5.5)^2 + 11(5.5) - 10 = 20.25$$

Full solution 3.2 (2/2)

Continuous maximum profit.

(b) Check the adjacent integers 5 and 6.

$$P(5) = -25 + 55 - 10 = 20$$

$$P(6) = -36 + 66 - 10 = 20$$

Both quantities maximise feasible integer profit.

(c) Factor the zero-profit equation: $P(q) = -(q - 1)(q - 10) = 0$, giving break-even quantities 1 and 10. Strictly positive profit is $1 < q < 10$, hence whole numbers 2 through 9.