

Exercise walk-through

3.2

Integration

GAC Mathematics

1

The method

Method — Key knowledge: a definite integral can cancel contributions with opposite signs

An **indefinite integral** 不定积分 gives a family of antiderivatives with a **constant of integration** 积分常数 C . The power rule is $\int x^n dx = x^{n+1}/(n+1) + C$ for $n \neq -1$, on a suitable domain. For the reciprocal, $\int 1/x dx = \ln|x| + C$ on an interval not crossing zero.

A **definite integral** 定积分 is signed accumulation. When an antiderivative F is valid over the interval, $\int_a^b f(x) dx = F(b) - F(a)$. Geometric area is nonnegative, so split where a curve changes sign. Substitution reverses the chain rule and requires matching differential factors; definite-integral bounds must change consistently or return to the original variable.

Method — Worked example: negative and positive areas can cancel

Known: $f(x) = x$ from negative 1 to 1. Its definite integral is zero because the negative and positive contributions cancel.

$$I = \int_{-1}^1 x \, dx = \left[\frac{x^2}{2} \right]_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$$

Geometric area instead adds the magnitudes of the two triangular regions.

$$A = - \int_{-1}^0 x \, dx + \int_0^1 x \, dx = \frac{1}{2} + \frac{1}{2} = 1$$

Exercises 2.1·2.2

2.1 Find $\int (3x^2 - 4) dx$.

[2]

2.2 Evaluate $\int_0^2 (2x + 1) dx$.

[2]

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2.1 Find $\int (3x^2 - 4) dx$.

[2]

2.1

Use the power rule term by term.

$$F(x) = x^3 - 4x + C$$

Correct antiderivative including C.

2.2 Evaluate $\int_0^2 (2x + 1) dx$.

[2]

2.2

Use antiderivative $F(x) = x^2 + x$ and subtract the lower-bound value.

$$I = [x^2 + x]_0^2 = (2^2 + 2) - (0^2 + 0) = 6$$

Value 6.

Exercise 2.3

2.3 Find $\int 6x/(3x^2 + 2) dx$ using substitution.

[3]

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Let $u = 3x^2 + 2$, so $du = 6x dx$.

$$I = \int \frac{1}{u} du = \ln |u| + C$$

Logarithmic antiderivative. Since $3x^2 + 2 > 0$ for real x , $I = \ln(3x^2 + 2) + C$.

Exercise 2.4

2.4 Find both the signed integral $\int_{-1}^1 x \, dx$ and the geometric area between that line and

3 Exam-style questions

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The signed integral is $[x^2/2]_{-1}^1 = 0$. Split the geometric area at zero and reverse the sign of the lower region.

$$A = - \int_{-1}^0 x \, dx + \int_0^1 x \, dx = 1$$

Area 1.

Exercise 3.1

3.1 A particle has velocity $v(t) = 4 - t$ metres per second for $0 \leq t \leq 6$ seconds and starts at position 5 m. Find (a) its displacement, (b) its total distance travelled, and (c) its final position. [6]

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3.1

(a) Integrate velocity for signed displacement.

$$\Delta s = \int_0^6 v(t) dt = \left[4t - \frac{t^2}{2} \right]_0^6 = 6 \text{ m}$$

Displacement 6 m.

(b) Velocity changes sign at $t = 4$ s. Split there and add magnitudes.

$$D = \int_0^4 (4 - t) dt - \int_4^6 (4 - t) dt = 8 + 2 = 10 \text{ m}$$

(c) Final position $s_f = s_0 + \Delta s = 5 + 6 = 11$ m.

Exercise 3.2

3.2 Find the geometric area enclosed by $y = x$ and $y = x^2 - 2x$. (a) Find their intersection x-values. (b) Determine which curve is upper between them. (c) Form and evaluate an integral for the enclosed area. (d) Check the result by splitting your integral at the midpoint of the interval. [8]

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3.2

(a) Equate the curve values.

$$x = x^2 - 2x \implies x(x - 3) = 0$$

Intersections at x equal to 0 and 3.

(b) The difference is $x - (x^2 - 2x) = x(3 - x) > 0$ for $0 < x < 3$, so the line is upper.

(c) Integrate upper minus lower.

Full solution 3.2

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(b) The difference is $x - (x^2 - 2x) = x(3 - x) > 0$ for $0 < x < 3$, so the line is upper.

(c) Integrate upper minus lower.

$$A = \int_0^3 [x - (x^2 - 2x)] dx = \int_0^3 (3x - x^2) dx$$

$$A = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^3 = \frac{9}{2}$$

Correct antiderivative. Area $9/2$ square units.

(d) Each half has area $\int_0^{3/2} (3x - x^2) dx = 9/4$ and $\int_{3/2}^3 (3x - x^2) dx = 9/4$. Correct half integrals. Their sum is $9/2$, matching the result.