

Exercise walk-through

3.1

Differentiation

GAC Mathematics

1

The method

Method — Key knowledge: a derivative exists only when the limiting slope exists

A **derivative** 导数 is a limiting rate of change, $f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)]/h$, when that limit exists. For a differentiable curve it is the tangent gradient. A sharp corner can prevent differentiability. A dot over a variable normally denotes a time derivative, not an arbitrary derivative with respect to x .

Use the power, product, quotient and **chain rule** 链式法则 for the function's form and domain. The quotient rule needs a nonzero denominator; trigonometric derivative rules here use radians. A **stationary point** 驻点 has derivative zero. A positive second derivative at it gives a local minimum and a negative one a local maximum; a zero second derivative leaves that test inconclusive. Check derivative signs or other evidence.

Method — Worked example: a tangent slope differs from a secant slope

Known: $f(x) = x^2$. The secant between x equal to 1 and 2 has slope 3, but the derivative at x equal to 1 is 2.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

$$m_{tan} = f'(1) = 2(1) = 2$$

The average slope over a finite interval need not equal the instantaneous slope at either endpoint.

Exercises 2.1·2.2

2.1 Differentiate $y = 4x^3 - 6x^2 + 5$.

[2]

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2.1

Apply the power rule term by term; a constant has derivative zero.

$$\frac{dy}{dx} = 12x^2 - 12x$$

Correct derivative.

2.2 Differentiate $y = (2x + 1)^3$.

[2]

2.2

Use the chain rule, including inner derivative 2.

$$\frac{dy}{dx} = 3(2x + 1)^2(2) = 6(2x + 1)^2$$

Correct derivative.

Exercise 2.3

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[3]

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Use the product rule.

$$y' = u'v + uv'$$

$$y' = 2x \sin x + x^2 \cos x$$

Correct first product contribution. Correct second contribution and their sum.

Exercise 2.4

2.4 For $y = (2x + 1)/(x - 1)$, state the excluded x-value and find the derivative. [3]

3 Exam-style questions

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Exclude $x = 1$ because the denominator is zero. Use the quotient rule.

$$y' = \frac{u'v - uv'}{v^2}$$

$$y' = \frac{2(x - 1) - (2x + 1)}{(x - 1)^2} = \frac{-3}{(x - 1)^2}$$

Correct derivative on the stated domain.

Exercise 3.1

3.1 Use difference quotients to examine $f(x) = |x|$ at x equal to 0. (a) Find the right-hand and left-hand limiting slopes. (b) Decide whether a derivative exists at 0. (c) Give the derivative for x greater than 0 and x less than 0. [6]

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3.1

At zero, the difference quotient is $|h|/h$. For $h > 0$ it equals 1, so the right-hand limit is 1. For $h < 0$ it equals negative 1, so the left-hand limit is negative 1. The two one-sided limits differ, so no derivative exists at zero. For positive x , $f'(x) = 1$. For negative x , $f'(x) = -1$. The function itself is still continuous at zero.

Exercise 3.2

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Differentiate.

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

Correct factorisation. Stationary x-values are 1 and 3.

$$f(1) = 1 - 6 + 9 + 1 = 5$$

$$f(3) = 27 - 54 + 27 + 1 = 1$$

Points (1, 5) and (3, 1). Use the second derivative to classify them.

$$f''(x) = 6x - 12$$

At 1 it is negative 6, giving a local maximum; at 3 it is 6, giving a local minimum. Both classifications correct. The first derivative is positive for $x < 1$ and $x > 3$, so increasing on those intervals. It is negative for $1 < x < 3$, so decreasing there.