

Exercise walk-through

2.8

Distributions – The Normal Distribution

GAC Mathematics

1

The method

Method — Key knowledge: normal probabilities depend on the model and its parameters

A **normal distribution** 正态分布 is a continuous symmetric bell-shaped model with mean μ and positive **standard deviation** 标准差 σ . Standardising gives a **z-score** 标准分, $z = (x - \mu) / \sigma$. Total area is 1; individual exact values have zero probability in this continuous model.

Use the supplied standard-normal cumulative values $\Phi(z) = P(Z \leq z)$: $\Phi(0) = 0.5000$, $\Phi(1) = 0.8413$, $\Phi(1.5) = 0.9332$, $\Phi(2) = 0.9772$. Symmetry gives $\Phi(-z) = 1 - \Phi(z)$. Interval probability is upper cumulative area minus lower cumulative area. These probabilities require a stated normal assumption; a mean and standard deviation alone do not prove that real data are normal.

Method — Worked example: a threshold is an area beyond a standardised value

Known: an invented measurement model has $\mu = 50$ and $\sigma = 10$. For values above 60, standardise the threshold.

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 50}{10} = 1$$

Use the complement of the supplied cumulative area.

$$P(X > 60) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

This is a model probability, not a guarantee that exactly 15.87% of a finite sample exceeds 60.

Exercises 2.1-2.2

2.1 A normal model has mean 100 and standard deviation 15. Find the z-score of 130.[2]

2.2 Using the supplied cumulative values, find the probability that a standard-normal value is greater than 1. [2]

Exercises 2.1-2.2

2.1 A normal model has mean 100 and standard deviation 15. Find the z-score of 130.[2]

2.1

Standardise using the model mean and standard deviation.

$$z = \frac{x - \mu}{\sigma} = \frac{130 - 100}{15} = 2$$

Z-score 2.

2.2 Using the supplied cumulative values, find the probability that a standard-normal value is greater than 1. [2]

2.2

Use the complement of the area at or below 1.

$$P(Z > 1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

Probability 0.1587.

Exercise 2.3

2.3 Use symmetry and the supplied values to find the probability that a standard-normal value lies between negative 1 and 1. [3]

Exercise 2.3

2.3 Use symmetry and the supplied values to find the probability that a standard-normal value lies between negative 1 and 1. [3]

2.3

Symmetry gives $\Phi(-1) = 1 - 0.8413 = 0.1587$. Subtract cumulative areas.

$$P(-1 < Z < 1) = \Phi(1) - \Phi(-1) = 0.8413 - 0.1587 = 0.6826$$

Probability 0.6826.

Exercise 2.4

2.4 A normal model has mean 70 and standard deviation 8. Find the raw values corresponding to z-scores negative 1 and 1.5. Give the probability above the latter value. [3]

Exercise 2.4

2.4 A normal model has mean 70 and standard deviation 8. Find the raw values corresponding to z-scores negative 1 and 1.5. Give the probability above the latter value. [3]

2.4

Invert standardisation: $x = \mu + z\sigma$.

$$x_1 = 70 + (-1)(8) = 62$$

Lower raw value.

$$x_2 = 70 + 1.5(8) = 82$$

Upper raw value.

$$P(X > 82) = 1 - \Phi(1.5) = 1 - 0.9332 = 0.0668$$

Tail probability.

Exercise 3.1

3.1 A stated normal model for completion time has mean 40 minutes and standard deviation 5 minutes. (a) Find the probability of completion between 35 and 50 minutes. (b) For 200 independently modelled tasks, find the expected number exceeding 50 minutes. Explain why that expectation is not a guaranteed observed count. [6]

Exercise 3.1

3.1 A stated normal model for completion time has mean 40 minutes and standard deviation 5 minutes. (a) Find the probability of completion between 35 and 50 minutes. (b) For 200 independently modelled tasks, find the expected number exceeding 50 minutes. Explain why that expectation is not a guaranteed observed count. [6]

3.1

(a) Standardise both bounds.

$$z_L = \frac{x_L - \mu}{\sigma} = \frac{35 - 40}{5} = -1$$

Both bounds correct. Subtract cumulative areas.

$$p = \Phi(2) - \Phi(-1) = 0.9772 - 0.1587 = 0.8185$$

Interval probability.

(b) Tail probability is $1 - 0.9772 = 0.0228$, so expected count $E = Np = 200(0.0228) = 4.56$. The random observed count is an integer and can vary; 4.56 is an expectation over repeated modelled samples.

Exercise 3.2

3.2 Two stated normal score models are A with mean 60 and standard deviation 10, and B with mean 75 and standard deviation 5. A learner scores 80 on A and 85 on B. (a) Find both z-scores and compare relative position. (b) Use the supplied table to find each upper-tail probability. (c) For 500 modelled scores on each scale, find each expected count above those thresholds. (d) Explain two limits of using these calculations to claim equal ability on different tests. [8]

Exercise 3.2

3.2 Two stated normal score models are A with mean 60 and standard deviation 10, and B with mean 75 and standard deviation 5. A learner scores 80 on A and 85 on B. (a) Find both z-scores and compare relative position. (b) Use the supplied table to find each upper-tail probability. (c) For 500 modelled scores on each scale, find each expected count above those thresholds. (d) Explain two limits of using these calculations to claim equal ability on different tests. [8]

3.2

(a) Standardise relative to each model.

$$z_A = \frac{80 - 60}{10} = 2$$

A z-score.

Full solution 3.2

3.2

(a) Standardise relative to each model.

$$z_A = \frac{80 - 60}{10} = 2$$

A z-score.

$$z_B = \frac{85 - 75}{5} = 2$$

B z-score. Both scores are two model standard deviations above their respective means.

(b) For each model, $p = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$.

(c) Expected count on each scale is $E = Np = 500(0.0228) = 11.4$.

(d) Normality and representative parameters must be justified for the actual data. Equal standardised relative position does not establish equivalent test content, difficulty, reliability or learner ability; separate evidence is needed.