

Exercise walk-through

1.6

Exponential and Logarithmic Functions

GAC Mathematics

1

The method

Method — Key knowledge: a logarithm finds the exponent, subject to its domain

An **exponent** 指数 tells a power. For positive a , $a^m a^n = a^{m+n}$, $a^m / a^n = a^{m-n}$ and $(a^m)^n = a^{mn}$. A zero power is 1 for a nonzero base. A negative power is a reciprocal, not a negative result.

A **logarithm** 对数 answers an exponent question: $\log_b x = y$ means $b^y = x$, with $b > 0$, $b \neq 1$, and $x > 0$ for real logarithms. Product, quotient and power laws do not give a rule for the logarithm of a sum. Natural logarithms have base e .

A model $Q(t) = Q_0 a^t$ has growth for $a > 1$ and **exponential decay** 指数衰减 for $0 < a < 1$. State the time unit and whether time is continuous or restricted to whole periods. The basic exponential graph has y-intercept 1 and stays above the x-axis; $\log_b x$ has x-intercept 1 and vertical asymptote $x = 0$.

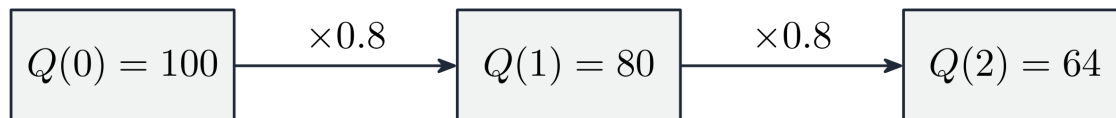
Method — Worked example: repeated reductions multiply rather than subtract a fixed amount

Known: an invented quantity starts at 100 units and decreases by 20% per whole period. The retained proportion is 0.8, so use a multiplicative model.

$$Q(n) = Q_0(1 - r)^n$$

$$Q(2) = 100(1 - 0.20)^2 = 64$$

The first decrease is 20 units and the second is 16. A fixed subtraction of 20 each period would be a different model.



equal time periods, changing absolute reductions

Exercises 2.1·2.2

2.1 Simplify $2^3 \times 2^{-5}$ to a single power and then to a fraction.

[2]

2.2 Solve $\log_3 x = 4$ in the real numbers.

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Exercises 2.1·2.2

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[2]

2.1

Same-base products add exponents.

$$E = 2^{3+(-5)} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

Correct power and fraction.

2.2 Solve $\log_3 x = 4$ in the real numbers.

[2]

2.2

Convert the logarithmic statement to exponential form.

$$x = 3^4 = 81$$

Positive solution 81.

Exercise 2.3

2.3 Solve $5 \times 2^t = 40$. Sketch $y = 2^t$, labelling its y-intercept and horizontal asymptote.[3]

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Divide both sides by 5.

$$2^t = \frac{40}{5} = 8 = 2^3$$

Thus $t = 3$. An increasing exponential curve through $(0, 1)$, approaching but never meeting $y = 0$ as t decreases, earns the final point. Label the y-intercept and asymptote.

Exercise 2.4

2.4 Solve $\ln(x - 1) + \ln 2 = \ln 10$. State the domain restriction. Sketch $y = \ln x$, labelling its x-intercept and vertical asymptote. [3]

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The equation requires $x - 1 > 0$, hence $x > 1$. Combine logarithms of positive factors.

$$\ln(2(x - 1)) = \ln 10 \implies 2(x - 1) = 10$$

$$x = 1 + \frac{10}{2} = 6$$

Solution 6 satisfies $x > 1$, and $\ln 5 + \ln 2 = \ln 10$. For the separate sketch, draw an increasing logarithmic curve on $x > 0$ through $(1, 0)$ with vertical asymptote $x = 0$; label both features.

Exercise 3.1

3.1 An account starts with 500 units and grows by 4% at the end of each year, with no deposits, withdrawals or charges. (a) Write a model for the value after n whole years. (b) Find the value after 3 years to 2 decimal places. (c) Find the first whole year at which its value is greater than 600 units. [6]

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3.1

(a) Each year's multiplier is 1.04.

$$A(n) = 500(1.04)^n$$

Correct model.

(b) Substitute three completed periods.

Full solution 3.1

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$$A(n) = 500(1.04)^n$$

Correct model.

(b) Substitute three completed periods.

$$A(3) = 500(1.04)^3 = 562.432 \approx 562.43$$

Correct rounded value.

(c) The strict threshold is $500(1.04)^n > 600$. Divide by 500 and use the increasing natural logarithm.

$$n > \frac{\ln(600/500)}{\ln 1.04} \approx 4.6486$$

Correct threshold. First whole year is 5: $A(4) \approx 584.93 < 600$ and $A(5) \approx 608.33 > 600$.

Exercise 3.2

3.2 A teaching model is $Q(t) = 200(0.85)^t$, where t is time in hours and may take any real value at least zero. (a) State the initial quantity and hourly percentage decrease. (b) Find $Q(4)$ to 3 significant figures. (c) Find when the quantity first reaches 100 units, to 3 significant figures. (d) Under the same model sampled only after whole hours, find the first sampled value at or below 100. Explain whether the model ever reaches zero at a finite time. [8]

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- (a) $Q(0) = 200$ units. Retained proportion is 0.85, so the hourly decrease is 15%.
- (b) Substitute four hours.

$$Q(4) = 200(0.85)^4 = 104.40125 \approx 104 \text{ units}$$

Correct 3 significant figures.

Full solution 3.2

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(b) Substitute four hours.

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Correct 3 significant figures.

(c) Set the quantity equal to the target and take logarithms.

$$0.85^t = \frac{100}{200} = 0.5$$

$$t = \frac{\ln(100/200)}{\ln 0.85} \approx 4.27 \text{ hours}$$

Correct continuous time.

(d) First qualifying sample is at 5 hours because $Q(4) > 100$ and $Q(5) \approx 88.74 < 100$. For every finite t the positive base raised to t stays positive; the model approaches zero but never reaches it.