GAC024 Discrete Mathematics

GAC024-aligned discrete mathematics practice; the centre's brief sets the actual assessment.

What this module is, and how it is marked

- use sets, Venn diagrams, relations and functions;
- convert between **decimal**, binary and **hexadecimal**;
- use Boolean logic, truth tables and **logic gates**;

What this module is, and how it is marked (2)

- trace algorithms and compare their **efficiency**;
- model problems with graphs and find shortest paths.

Sets, relations and functions

- A **set** is a collection of distinct objects. Order and repetition do not matter.
- **Union** $A \cup B$ is everything in either; **intersection** $A \cap B$ is what is in
- A **subset** has all its elements inside another set.

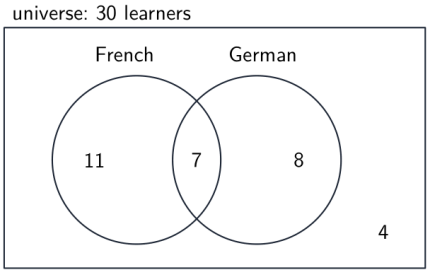
$$|F \cup G| = |F| + |G| - |F \cap G| = 18 + 15 - 7 = 26$$

Sets, relations and functions (2)

- A **relation** pairs elements of two sets. A **function** is a **relation** in which each input has exactly one output. "Each student $\rightarrow$ their birthday" is a **function** (two students may share a birthday); "each birthday $\rightarrow$ the students born on it" is not, because one input can have several outputs.
- A **Venn diagram** , introduced by the English logician John Venn in 1880, turns a set problem into a picture. Check that the regions add up to the total.

$$N_{neither} = |U| - |F \cup G| = 30 - 26 = 4$$

Figure: Sets, relations and functions



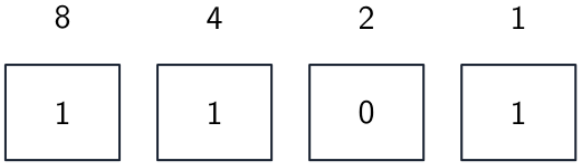
- A class universe of 30 is divided into French-only 11, both 7, German-only 8 and neither 4.
- Match each part of the diagram to the text.

Counting systems

- A **number base** b uses the digits 0 to $b-1$, and each place is worth a power of b. **Decimal** uses ten, **binary** two, **hexadecimal** sixteen (digits 0–9 then A–F).
- Every digit's value is its **place value**: in binary the places are 1, 2, 4, 8, 16 and so on.
- **Hexadecimal** is shorthand for binary: one hex digit is exactly four bits. Web colours use it: '#FF8000' means red 255, green 128, blue 0, a bright orange.

$$V = \sum d_i 2^i$$

Figure: Counting systems



place weights above binary digits

- The binary digits 1101 are aligned with place weights 8, 4, 2 and 1.
- Match each part of the diagram to the text.

Binary applications

- **Binary arithmetic** adds like **decimal**, carrying at 2 instead of at 10.
- A **bit** is one binary digit; a **byte** is eight.
- **Boolean algebra** works on true and false with AND, OR and NOT.

Binary applications (2)

- A **truth table** lists every combination of inputs and the output for each. Two expressions are equivalent if their truth tables match in every row.
- **Logic gates** carry out these operations in electronics, and a **logic circuit** connects them. George Boole published his algebra of logic in 1854; in 1937 Claude Shannon showed that it describes switching circuits exactly, which is the basis of every digital computer.

Figure: Binary applications



- Inputs A and B enter an OR-labelled block, whose output enters a NOT-labelled block to give Y.
- Match each part of the diagram to the text.

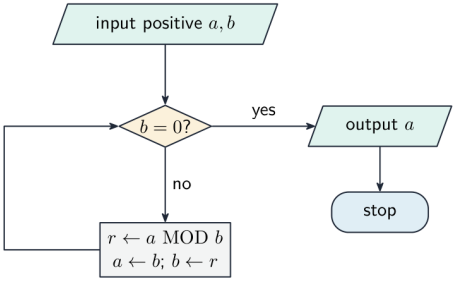
Algorithms

- An **algorithm** is a list of clear steps that solves a task and always finishes. The word comes from the name of al-Khwarizmi, a ninth-century mathematician in Baghdad.
- A **flowchart** draws it: a decision is a diamond, a process a rectangle.
- **Pseudocode** writes the steps in structured English, without a particular programming language.

Algorithms (2)

- **Tracing** an **algorithm** means following it by hand in a table with one column per variable and one row per step.
- **Efficiency** matters. A linear search checks items one by one, up to n checks. A binary search of a sorted list halves the range each time: a sorted list of 1,000,000 names needs at most 20 checks, because $2^{20} > 1{,}000{,}000$.

Figure: Algorithms



- A Euclidean-algorithm **flowchart** tests b equal to zero, otherwise computes a remainder and updates the pair before returning to the test.
- Match each part of the diagram to the text.

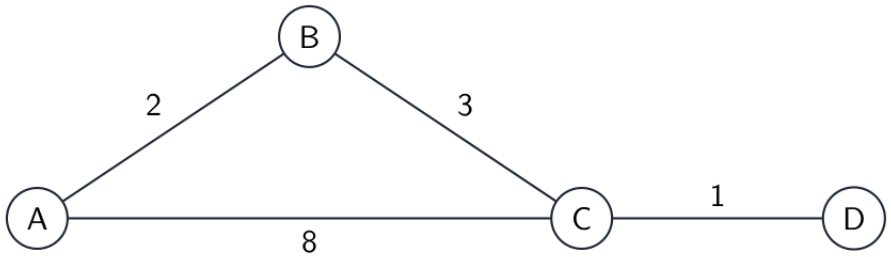
Graphs and networks

- A **graph** is a set of **vertices** joined by **edges**. It models anything with
- The **degree** of a vertex is the number of **edges** that meet it. Every edge has two ends, so the degrees always add up to twice the number of edges. This is the fact behind Euler's Königsberg proof.
- A **tree** is a connected **graph** with no cycles; a **tree** with n vertices has $n - 1$ edges. A family tree and a computer's folder structure are trees.

Graphs and networks (2)

- A **shortest path** problem asks for the route with the smallest total weight (distance, time or cost), not the fewest **edges**. Edsger Dijkstra designed his famous **algorithm** for it in 1956; navigation apps still use versions of it.
- A **minimum spanning tree** connects every vertex with the smallest total edge weight and no cycles, the cheapest way to link towns with cable.

Figure: Graphs and networks



- An undirected network joins A to B with weight 2, B to C with 3, A to C with 8 and C to D with 1.
- Match each part of the diagram to the text.

Sources for this lesson

- GAC024 Discrete Mathematics course outline.
- L. Euler, "Solutio problematis ad geometriam situs pertinentis" (1736), the Königsberg bridges.
- G. Boole, *An Investigation of the Laws of Thought* (1854); C. E. Shannon, "A Symbolic Analysis of Relay and Switching Circuits" (1938).

Sources for this lesson (2)

- E. W. Dijkstra, "A Note on Two Problems in Connexion with Graphs", *Numerische Mathematik* 1 (1959).
- Euclid, *Elements*, Book VII, Propositions 1–2.

Common errors

- Adding the two set totals without subtracting the overlap: $|A \cup B| = |A| + |B| - |A \cap B|$.
- Calling a **relation** a **function** when one input has two outputs, or rejecting one because two inputs share an output.
- Reading binary place values from the left: the rightmost **bit** is worth 1.

Common errors (2)

- Grouping binary into fours from the left when converting to **hexadecimal**; group from the right.
- Forgetting that a fixed-width result keeps only the low bits when the question states wraparound.
- Treating OR as exclusive: inclusive OR is true when both inputs are true.

Common errors (3)

- Changing only one part when applying De Morgan: $\neg(A \vee B) = \neg A \wedge \neg B$; the operator flips too.
- Choosing the path with fewest **edges** instead of the smallest total weight.

Key terms

- **set**: a collection of distinct objects.
- **intersection**: the elements that are in both sets.
- **function**: a **relation** in which each input has exactly one output.

Key terms (2)

- **place value**: what a digit is worth because of its position.
- **truth table**: a table of every input combination and its output.
- **algorithm**: a finite list of clear steps that solves a task.

Key terms (3)

- **tracing**: following an **algorithm** by hand, step by step.
- **degree**: the number of **edges** meeting at a vertex.
- **tree**: a connected **graph** with no cycles.
- **shortest path**: the route between two vertices with the smallest total weight.

Check yourself

- Answer each question before reading the answers.
- 1.
- In a group of 40 students, 25 play football, 18 play basketball and 8 play both.
- How many play neither?