GAC016 Mathematics III: Calculus

GAC016-aligned calculus practice; the centre's brief sets the actual assessment and grading.

What this module is, and how it is marked

- find derivatives from first principles and with the power, product, quotient and chain rules;
- find and classify **stationary points**;
- integrate, find definite integrals and areas, and use substitution;
- solve **optimisation**, connected-rates and area-between-curves problems.

Differentiation

- The **power rule** : if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$.
- The **product rule** : $(uv)' = u'v + uv'$.
- The **quotient rule** , where $v \neq 0$: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$.

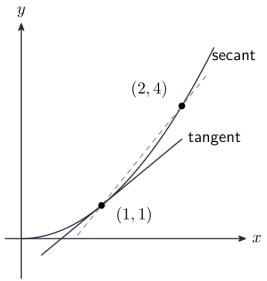
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation (2)

- The **chain rule** : if $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x)$.
- **Stationary points** occur where $f'(x) = 0$. The **second derivative** then

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

Figure: Differentiation



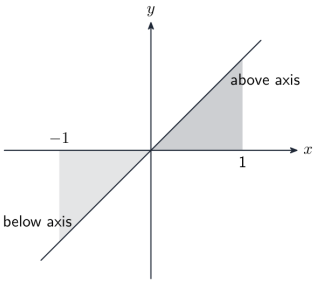
- The curve y equals x squared has a tangent at (1,1) and a dashed secant joining (1,1) to (2,4).
- Read the axes and units first.

Integration

- An **indefinite integral** has no limits and needs the **constant of integration**
- A **definite integral** has limits and gives a number:
- **Integration by substitution** reverses the **chain rule**.
- △ Forgetting $+c$ on an **indefinite integral** loses marks on answers that are otherwise correct.

$$I = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

Figure: Integration



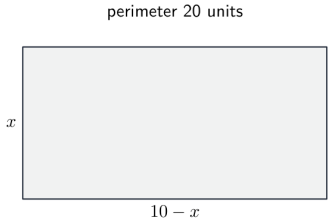
- The line y equals x crosses the axis at zero, with shaded regions on both sides over negative 1 to 1.
- Read the axes and units first.

Advanced applications

- **Optimisation** finds the largest or smallest value of a quantity: write the quantity
- **Rates of change** chain together: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$.
- **Area between two curves** is $\displaystyle\int_a^b (f(x) - g(x))\,dx$ where f is the upper curve on that whole interval. Split at crossings if their order changes.
- **Marginal cost and revenue** are the derivatives of cost and revenue: approximately the extra cost or revenue of one more unit.

$$A(x) = x(10 - x)$$

Figure: Advanced applications



- A rectangle has width x and length 10 minus x under a fixed perimeter of 20 units.
- Match each part of the diagram to the text.

Sources for this lesson

- GAC016 Mathematics III course outline.
- IAAF (now World Athletics), biomechanics report on the 2009 Berlin World Championships 100 m final.
- I. Newton, *Philosophiae Naturalis Principia Mathematica* (1687); G. W. Leibniz, "Nova Methodus pro Maximis et Minimis" (1684).

Common errors

- Using the secant slope when the question asks for an instantaneous rate, or the reverse.
- Forgetting the inner derivative in the **chain rule**: $\frac{d}{dx}(3x-1)^4 \neq 4(3x-1)^3$.
- Writing $(uv)' = u'v'$. The **product rule** has two terms.

Common errors (2)

- Swapping the numerator order in the **quotient rule**: it is $u'v - uv'$, not $uv' - u'v$.
- Classifying a stationary point with $f'' = 0$ as a point of inflection without checking the sign of f' on both sides.
- Dropping $+c$, or treating a **definite integral** of a sign-changing function as an area.

Common errors (3)

- Not checking the ends of the feasible domain in an **optimisation** question.
- Leaving the answer to a related-rates question without units or a sign that shows increase or decrease.

Key terms

- **derivative**: the rate of change of a function; the gradient of its tangent.
- **first principles**: finding a derivative from the limit of chord gradients.
- **chain rule**: the rule for differentiating a function of a function.

Key terms (2)

- **stationary point**: a point where the derivative is zero.
- **second derivative**: the derivative of the derivative, used to classify **stationary points**.
- **indefinite integral**: an antiderivative, written with $+c$.

Key terms (3)

- **definite integral**: an integral with limits, giving a signed total.
- **optimisation**: finding the largest or smallest value of a quantity.
- **marginal cost**: the derivative of cost; about the cost of one more unit.

Check yourself

- Answer each question before reading the answers.
- 1.
- Differentiate (a) $y = 5x^4 - \frac{3}{x} + 2\sqrt{x}$; (b) $y = (2x+3)^5$; (c) $y = x e^{2x}$; (d) $y = \frac{x^2}{x+1}$. 2.
- Use first principles to show that the derivative of $f(x) = 3x^2$ is $6x$. 3.