GAC004 Mathematics I: Fundamentals

GAC004-aligned fundamentals practice; the centre's brief sets the actual assessment and grading.

What this module is, and how it is marked

- use the vocabulary of number, percentages, significant figures and standard form;
- expand, factorise and solve linear and quadratic equations, inequalities and simultaneous equations;
- find gradients, midpoints and distances, and read graphs;

What this module is, and how it is marked (2)

- calculate angles, areas and volumes, and use the trigonometric ratios and rules;
- use exponential and logarithmic functions for growth, decay and scales.

Terminology and arithmetic review

- A **factor** divides a number exactly; a **multiple** is the number times a whole number. 6 is a factor of 24, and 24 is a multiple of 6.
- The **highest common factor (HCF)** and **lowest common multiple (LCM)** come from prime factors: $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, so the HCF is $2^2 \times 3 = 12$ and the LCM is $2^3 \times 3^2 = 72$.
- A **percentage** is a fraction out of 100. An increase of 15% multiplies by 1.15; a decrease of 15% multiplies by 0.85. To reverse a change, divide by the multiplier.

$$3 + 4 \times 2^2 = 3 + 4 \times 4 = 3 + 16 = 19$$

Terminology and arithmetic review (2)

- **Significant figures** show how precisely a value is given. China's population at the end of 2023 was 1,409,670,000, or 1.41×10^9 to three significant figures.
- **Standard form** writes a number as $a \times 10^n$ with $1 \leq a < 10$ and n an integer. The speed of light is 3.00×10^8 m/s; the mass of an electron is 9.11×10^{-31} kg.

$$P_1 = 80 \times 0.8 = ¥64 \qquad P_2 = 64 \times 1.2 = ¥76.80$$

Figure: Terminology and arithmetic review



each multiplier uses the current amount

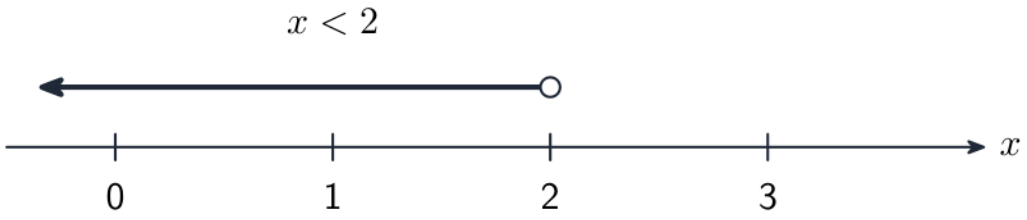
- ¥80 falls to ¥64, then rises to ¥76.80: each percentage uses the price at that stage
- Match each part of the diagram to the text.

Algebra I: introductory algebra

- To solve a **linear equation** , do the same operation to both sides until the letter stands alone.
- A **quadratic equation** has the form $ax^2 + bx + c = 0$ with $a \neq 0$. Factorise when you can, or use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** are two equations in two unknowns, solved by substitution or elimination. Two straight lines meet at one point, or never (parallel), or everywhere (the same line).
- △ An inequality **reverses** when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

$$(x+3)(x-5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

Figure: Algebra I: introductory algebra



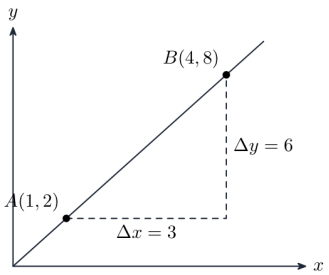
- A number line shows x less than 2, with an open circle at 2.
- Match each part of the diagram to the text.

Algebra: graphs and coordinate geometry

- Lines are **parallel** when their gradients are equal, and **perpendicular** when their gradients multiply to -1.
- The **midpoint** of a segment is the average of the endpoints; the **distance** between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graph is a **parabola**. Its **vertex** is the turning point; its **roots** are where it meets the x-axis. The path of a thrown ball is close to a parabola.
- To solve two equations **graphically**, draw both and read where they cross; check the answer algebraically, because a drawing is only approximate.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Figure: Algebra: graphs and coordinate geometry



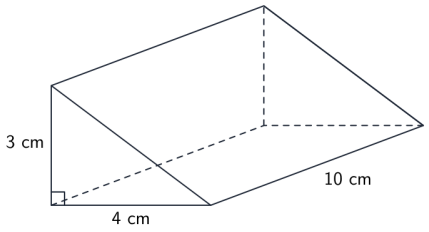
- The line passes through A at (1,2) and B at (4,8), with dashed coordinate-change guides.
- Match each part of the diagram to the text.

Geometry: plane, solid and Euclidean geometry

- **Congruent** shapes are identical in size and shape; **similar** shapes have the same shape, with all lengths in one ratio.
- **Pythagoras' theorem** : in a right-angled triangle, $a^2 + b^2 = c^2$, where c is the **hypotenuse** . Ancient Chinese mathematicians knew it as the *gougu* rule; the *Zhoubi Suanjing* shows the 3, 4, 5 triangle.
- **Area** is measured in square units and **volume** in cubic units. A cylinder has volume $\pi r^2 h$; a sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.
- △ Scaling is not linear. Double every length of a solid and its area is multiplied by 4, its volume by 8. Galileo used this in 1638 to explain why a giant animal needs much thicker bones: its weight grows with volume, but bone strength grows only with cross-sectional area.

$$A = \tfrac{1}{2}(3)(4) = 6 \text{ cm}^2 \qquad V = A \times L = 6 \times 10 = 60 \text{ cm}^3$$

Figure: Geometry: plane, solid and Euclidean geometry



not to scale

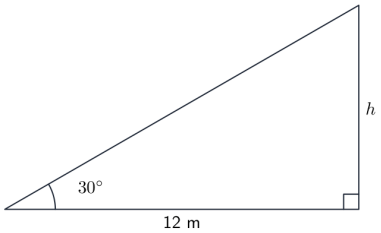
- A triangular prism has a right-triangle cross-section with perpendicular sides 3 and 4 cm and length 10 cm; perspective is not to scale.
- Match each part of the diagram to the text.

Trigonometry

- An angle can be measured in **degrees** or **radians** ; π radians $= 180°$. Check your calculator's mode before every question.
- The **unit circle** extends sine and cosine beyond $90°$; they repeat every $360°$, so they are **periodic** .
- A **bearing** is measured clockwise from north and written with three figures, such as $075°$.
- **Angles of elevation and depression** are measured from the horizontal, up and down. Add the observer's eye height if the question asks for height above the ground.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Figure: Trigonometry



not to scale

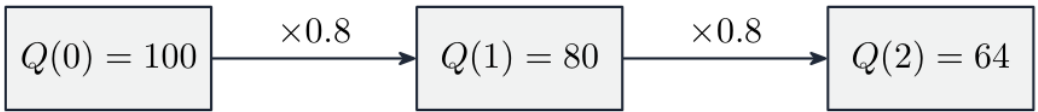
- A right triangle has a 12 m horizontal base and an angle of elevation of 30 degrees; the unknown height is opposite the angle.
- Match each part of the diagram to the text.

Exponential and logarithmic functions

- The laws turn multiplication into addition, which solves equations with the unknown in the exponent.
- **Natural logarithms** use the base $e \approx 2.718$ and are written $\ln$.
- **Compound interest** is exponential: $A = P(1 + r)^n$.
- **Exponential decay** has a base between 0 and 1. Carbon-14 has a half-life of about 5,730 years, so after t years a fraction $\left(\frac{1}{2}\right)^{t/5730}$ remains; this is how archaeologists date wood and bone.

$$\log(mn) = \log m + \log n, \qquad \log\left(\frac{m}{n}\right) = \log m - \log n, \qquad \log(m^k) = k \log m$$

Figure: Exponential and logarithmic functions



equal time periods, changing absolute reductions

- A model starting at 100 is multiplied by 0.8 each period, giving 80 then 64.
- Match each part of the diagram to the text.

Sources for this lesson

- GAC004 Mathematics I course outline.
- Shanghai Stock Exchange, Composite Index closing values, 16 October 2007 and 28 October 2008.
- National Bureau of Statistics of China, population at the end of 2023.

Sources for this lesson (2)

- Galileo Galilei, *Two New Sciences* (1638), on scaling and the strength of bones.
- Cleomedes, *On the Heavens*, the ancient account of Eratosthenes' measurement.
- China Earthquake Administration, magnitude of the 2008 Wenchuan earthquake.

Common errors

- Undoing a percentage change with the same percentage: a 20% fall then a 20% rise does not return to the start.
- Multiplying a minus sign into only the first term of a bracket: $-2(x-4) = -2x+8$.
- Keeping the inequality sign when multiplying or dividing by a negative number.

Common errors (2)

- Swapping rise and run, or mixing the order of the coordinates, in a gradient.
- Mixing area and volume units, or radius and diameter, in mensuration formulas.
- Using SOH CAH TOA in a triangle without a right angle; use the sine or cosine rule there.

Common errors (3)

- Leaving the calculator in radian mode for a degree question.
- Treating percentage growth as adding a fixed amount each period.

Key terms

- **integer**: a whole number, positive, negative or zero.
- **irrational number**: a number that cannot be written as a fraction of two integers.
- **significant figures**: the digits that show how precisely a number is given.

Key terms (2)

- **standard form**: a number written as $a \times 10^n$ with $1 \leq a < 10$.
- **coefficient**: the number multiplying a variable in a term.
- **identity**: an equation that is true for every value of the variable.

Key terms (3)

- **gradient**: the steepness of a line, rise divided by run.
- **similar**: having the same shape, with all lengths in the same ratio.
- **hypotenuse**: the longest side of a right-angled triangle, opposite the right angle.

Key terms (4)

- **bearing**: a direction measured clockwise from north, in three figures.
- **logarithm**: the power to which a base must be raised to give a number.
- **exponential decay**: a decrease by the same percentage in each period.

Check yourself

- Answer each question before reading the answers.
- 1.
- A price of ¥80 rises by 25% and the new price then falls by 20%.
- Find the final price and explain the result. 2.