

GAC016 Mathematics III: Calculus & Advanced Applications

GAC Mathematics

What this module is, and how it is marked

An average speed does not tell you every instantaneous speed, and a signed displacement does not always equal total distance. Calculus makes those distinctions precise. GAC016 develops **differentiation**, **integration** and their applications. Your centre's current brief determines assessment tasks and grading requirements; these original practice sheets do not establish a university credit decision or an official examination pattern.

State the independent variable, domain and units. The notation dy/dx is a derivative with respect to x ; a dot over y normally denotes a derivative with respect to time. Those notations describe the same operation only when their independent variables agree.

For applications, describe what a rate or accumulated value means in the stated model. A negative volume-change rate of 3 cubic centimetres per second means volume is falling at that rate, rather than that the volume itself is negative.

Differentiation

A **derivative** is a rate of change: how fast y changes as x changes. Geometrically it is the **gradient of the tangent** at a point.

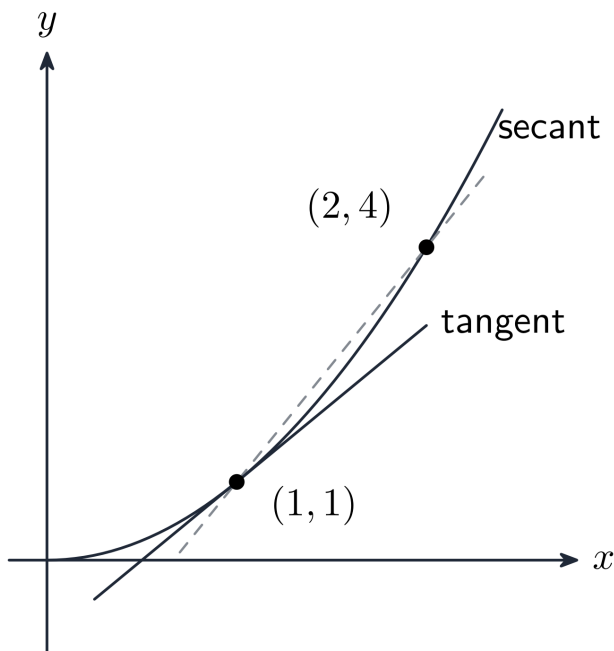
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

That limit is the definition. **Differentiation from first principles** uses the difference quotient for nonzero h and then takes its limit. Unequal one-sided limits, as for $|x|$ at zero, prevent a derivative even when the function is continuous.

- The **power rule**, on a domain where the power is differentiable: if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.
- The **product rule**: $(uv)' = u'v + uv'$.
- The **quotient rule**, where $v \neq 0$: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.
- The **chain rule**: if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.
- **Stationary points** occur where $f'(x) = 0$. The **second derivative** then classifies a stationary point when it is nonzero: negative gives a local **maximum** and positive a local **minimum**. A zero second

derivative is inconclusive; use derivative signs or other evidence. A stationary inflection can be neither an extremum.

Worked example. a tangent slope differs from a secant slope



Known: $f(x) = x^2$. The secant between x equal to 1 and 2 has slope 3, but the derivative at x equal to 1 is 2.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

$$m_{tan} = f'(1) = 2(1) = 2$$

The average slope over a finite interval need not equal the instantaneous slope at either endpoint.

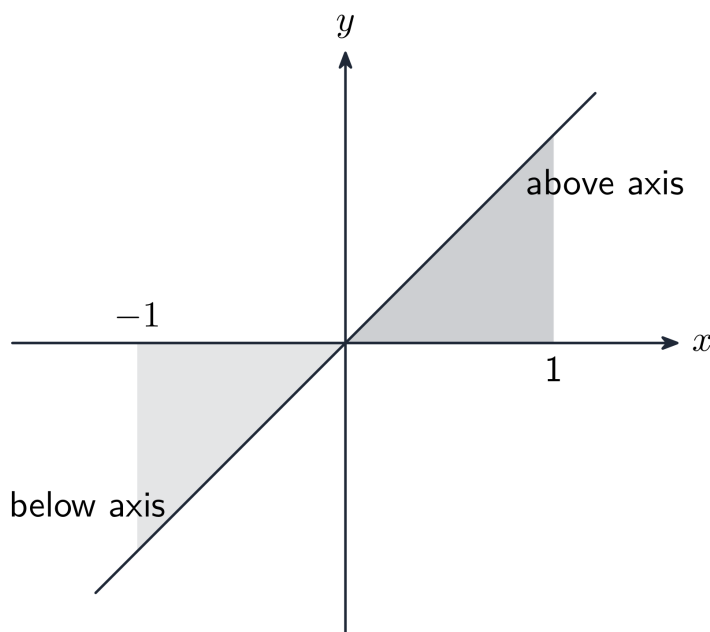
Use original practice sheet **3.1** to test the method, domain and interpretation against its solutions.

Integration

Indefinite integration finds antiderivatives. Definite integration measures signed accumulation, which can differ from geometric area. For a continuous integrand, the **fundamental theorem of calculus** connects a definite integral to the difference of antiderivative values at its limits.

- An **indefinite integral** has no limits and needs the **constant of integration** : $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** has limits and gives a number: $\int_a^b f(x) dx = F(b) - F(a)$.
- **Integration by substitution** reverses the chain rule.
- $\triangle$ Forgetting $+c$ on an indefinite integral is the single most frequent lost mark in the module, and it is lost on questions you have otherwise answered correctly.

| **Worked example.** negative and positive areas can cancel



Known: $f(x) = x$ from negative 1 to 1. Its definite integral is zero because the negative and positive contributions cancel.

$$I = \int_{-1}^1 x dx = \left[\frac{x^2}{2} \right]_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$$

Geometric area instead adds the magnitudes of the two triangular regions.

$$A = - \int_{-1}^0 x \, dx + \int_0^1 x \, dx = \frac{1}{2} + \frac{1}{2} = 1$$

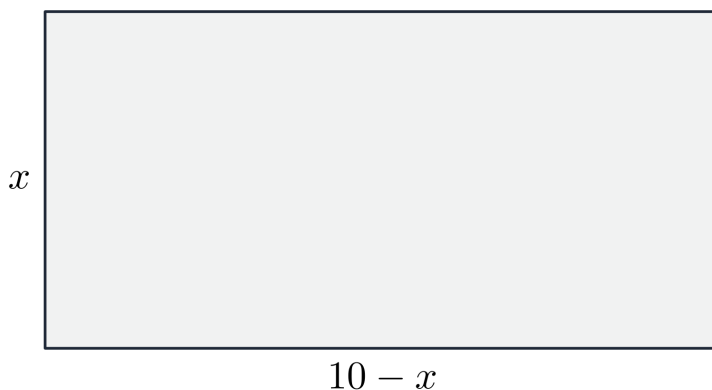
Use original practice sheet **3.2** to test the method, domain and interpretation against its solutions.

Advanced applications

- **Optimisation** finds the largest or smallest value of a quantity: write the quantity as a function of one variable on a stated feasible domain, differentiate, find candidates and justify the required optimum. Check included boundaries and integer constraints where relevant.
- **Rates of change** chain together: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$.
- **Area between two curves** is $\int_a^b (f(x) - g(x)) \, dx$ where f is the upper curve on that whole interval. Split at crossings if their order changes.
- **Marginal cost and revenue** are derivatives of stated continuous cost and revenue models. They are instantaneous rates, not automatically exact discrete-unit changes.

| **Worked example.** a fixed perimeter leaves one area variable

perimeter 20 units



not to scale

Known: a rectangle has perimeter 20, so length plus width is 10. Let width be x ; length is $10 - x$ and $0 < x < 10$.

$$A(x) = x(10 - x)$$

$$A'(x) = 10 - 2x$$

The stationary candidate is $x = 5$. Since $A''(x) = -2 < 0$ and the area approaches zero at both domain ends, this candidate is the global maximum. It is a 5 by 5 square with area 25.

Use original practice sheet **3.3** to test the method, domain and interpretation against its solutions.