

GAC004 Mathematics I: Fundamentals

GAC Mathematics

What this module is, and how it is marked

A price can fall by 20% and then rise by 20% without returning to its starting value. Mathematics I helps you explain such results with a method, units and a clear interpretation. The six units connect arithmetic, algebra, graphs, geometry, trigonometry and exponential models.

Your centre's current assessment brief is the authority for tasks, weights, permitted tools and deadlines. An **in-class test** is completed under stated classroom conditions. **Projects** can involve collecting data, calculating and reporting; an **examination** and **coursework** may have different instructions. Do not infer an official assessment pattern from these practice sheets.

A **terminology logbook** can record the English term, a meaning in your own words and a small example. Follow your centre's instructions if it is submitted or assessed.

Read the **command word** : *solve* asks for values satisfying a condition, *simplify* for an equivalent expression, and *justify* for a reason. Show a valid method so a reader can check how the result follows. The mark scheme for each task determines its marks.

The six accompanying practice sheets are original GAC004-aligned material, not official papers. Their solutions award marks for stated mathematical steps rather than for the length of a written report.

Terminology and arithmetic review

The vocabulary of number is assumed by every later unit.

An **integer** is a whole number, including negative whole numbers and zero. A **rational number** can be written as a fraction of two integers with a nonzero denominator; an **irrational number** cannot, and π and $\sqrt{2}$ are the standard examples. A **prime number** is a positive integer greater than 1 with exactly two positive factors, itself and 1.

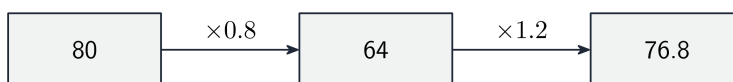
The **order of operations** fixes what a written expression means: brackets, then indices, then multiplication and division, then addition and subtraction, working left to right within a level.

$$3 + 4 \times 2^2 = 3 + 4 \times 4 = 3 + 16 = 19$$

- A **factor** divides a number exactly; a **multiple** is what you get by multiplying it. 6 is a factor of 24, and 24 is a multiple of 6.

- The **highest common factor (HCF)** and **lowest common multiple (LCM)** come from the prime factors: $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, so the HCF is $2^2 \times 3 = 12$ and the LCM is $2^3 \times 3^2 = 72$.
- A **percentage** is a fraction with denominator 100. An increase of 15% multiplies by 1.15; a decrease of 15% multiplies by 0.85. Reversing a percentage change means dividing, never subtracting the same percentage back.
- **Significant figures** are the digits retained to express a value at a stated precision. They do not alone establish the accuracy of a measurement. 0.004 072 to three significant figures is 0.004 07.
- **Standard form** writes a number as $a \times 10^n$ with $1 \leq |a| < 10$ for a nonzero number, with integer n. It is how a calculator shows a very large or very small answer.

Worked example. a decrease and increase use different bases



each multiplier uses the current amount

Known: an invented price of 80 units falls by 20%, then rises by 20%. Why use successive multipliers? Each change is calculated from the current price.

$$P_{new} = P_{old}(1 - r/100)$$

$$P_1 = 80(1 - 20/100) = 64$$

$$P_{new} = P_{old}(1 + r/100)$$

$$P_2 = 64(1 + 20/100) = 76.8$$

The price is not back at 80. The decrease was 16, while the later increase was 12.8, because the bases differ.

Continue with practice sheet **1.1**. Solve before opening the solutions, and check both the method and the final units.

Algebra I: introductory algebra

Algebra is arithmetic with letters standing for numbers, and it has its own vocabulary.

In $5x^2 - 3x + 7$, the whole thing is an **expression**; $5x^2$, $-3x$ and 7 are its **terms**; 5 is the **coefficient** of x^2 ; and 7 is a **constant**.

An **equation** says two expressions are equal and is solved for a value. An **identity** is true for every value in its domain. An **inequality** compares two expressions with $<$, $\leq$, $>$ or $\geq$.

To **expand** is to remove brackets; to **factorise** is to put them back.

$$(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

- Solving a **linear equation** means doing the same operation to both sides until the letter stands alone.
- A **quadratic equation** has the form $ax^2 + bx + c = 0$ with $a \neq 0$.
Factorise when suitable, or use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** are two equations in two unknowns, solved by substitution or by elimination. For two linear equations, a unique solution is their intersection. Parallel distinct lines have no solution; identical lines have infinitely many.
- $\triangle$ An inequality **reverses** when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

Worked example. reverse the inequality sign when dividing by a negative



Known: solve $7 - 3x > 1$. Subtract 7 from both sides, then divide by negative 3. The last operation reverses the inequality.

$$7 - 3x > 1 \implies -3x > -6$$

$$x < \frac{-6}{-3} = 2$$

The open circle excludes 2. Check $x = 1$: $7 - 3(1) = 4 > 1$; the boundary $x = 2$ gives equality and is excluded.

Continue with practice sheet **1.2**. Solve before opening the solutions, and check both the method and the final units.

Algebra: graphs and coordinate geometry

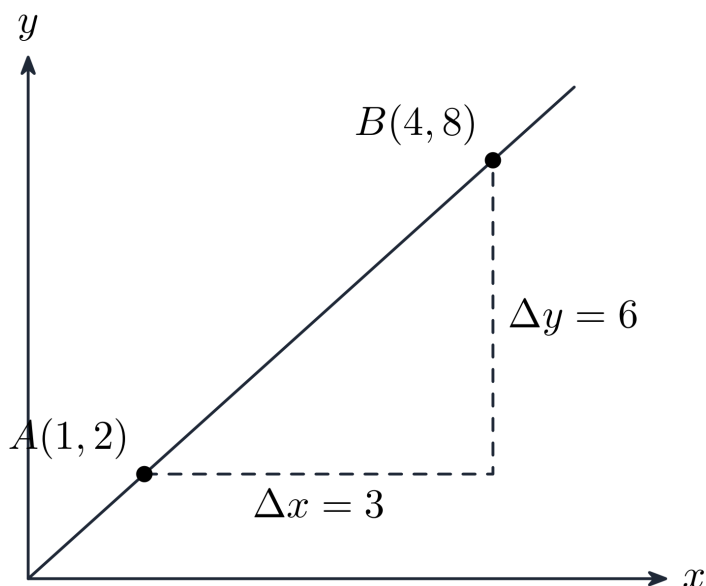
A graph turns an equation into a picture, and coordinate geometry measures that picture.

A nonvertical straight line can be written $y = mx + c$, where m is the **gradient** and c the **y-intercept**. The gradient is the rise divided by the run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Distinct nonvertical lines are **parallel** when their gradients are equal. Two nonvertical lines are **perpendicular** when their gradients multiply to -1 . Vertical lines have undefined gradient; a vertical and a horizontal line are perpendicular.
- The **midpoint** of a segment is the average of the endpoints, and the **distance** between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graphs as a **parabola**. Its **vertex** is the turning point, and the **roots** are the x-values where it meets the x -axis, the solutions of $y = 0$. A repeated root touches the axis without crossing; some quadratics have no real roots.
- To solve two equations **graphically**, draw both and read the coordinates of the intersection. The answer is only as accurate as the drawing, which is why an algebraic check matters.

Worked example. a rising line through two given points



Known: $A = (1, 2)$ and $B = (4, 8)$. Use gradient because the line is nonvertical.

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{8 - 2}{4 - 1} = 2$$

The line equation is $y = mx + c$. Substitute A to find the intercept.

$$c = y_A - mx_A = 2 - 2(1) = 0$$

Thus $y = 2x$. The diagram's horizontal change is 3 and vertical change is 6.

Continue with practice sheet **1.3**. Solve before opening the solutions, and check both the method and the final units.

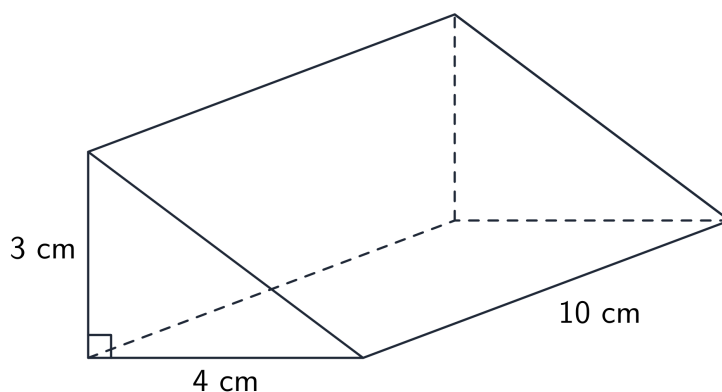
Geometry: plane, solid and Euclidean geometry

Plane geometry is about flat shapes, solid geometry about three-dimensional ones, and Euclidean geometry is the reasoning that connects them.

Angles on a straight line add to 180° , angles around a point to 360° , and the interior angles of a plane Euclidean triangle to 180° . In a simple **polygon** of n sides the interior angles add to $(n - 2) \times 180^\circ$.

- **Congruent** shapes are identical in size and shape; **similar** shapes have the same shape with all lengths in one ratio.
- **Pythagoras' theorem** holds in a right-angled triangle: $a^2 + b^2 = c^2$, where c is the **hypotenuse**.
- **Area** is measured in square units and **volume** in cubic units. A cylinder has volume $\pi r^2 h$; a sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.
- $\triangle$ Scaling is not linear. If every length of a solid is doubled, its area is multiplied by $2^2 = 4$ and its volume by $2^3 = 8$.

Worked example. a triangular prism has a constant cross-section



not to scale

Known: perpendicular triangle sides are 3 and 4 cm; prism length is 10 cm. Find the cross-sectional area, then multiply by the prism length.

$$A = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6 \text{ cm}^2$$

$$V = AL = 6(10) = 60 \text{ cm}^3$$

The sloping triangle side is not its perpendicular height. Volume is not the area of the triangular end.

Continue with practice sheet **1.4**. Solve before opening the solutions, and check both the method and the final units.

Trigonometry

Trigonometry connects the angles of a triangle to its sides.

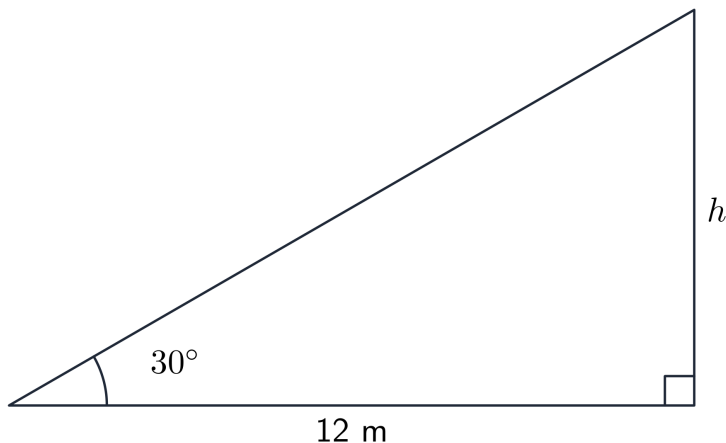
In a right-angled triangle, with θ one of the acute angles, the three ratios are $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$.

For a nondegenerate plane Euclidean triangle, the **sine rule** and the **cosine rule** apply:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad a^2 = b^2 + c^2 - 2bc \cos A$$

- An angle can be measured in **degrees** or in **radians**, where π radians is 180° . Check which mode your calculator is in before every question.
- The **unit circle** extends the ratios beyond 90° and explains why $\sin$ and $\cos$ repeat every 360° : they are **periodic**.
- A **bearing** is measured clockwise from north and always written with three figures, such as 075° .
- **Angles of elevation and depression** are measured from the horizontal, upwards and downwards respectively. Add the observer or instrument height if the answer needs height above the ground. With side-side-angle data, the sine rule may allow two triangles: check both supplementary angles against the remaining angle sum.

Worked example. height from a horizontal distance and angle



not to scale

Known: horizontal distance $d = 12$ m and elevation $\theta = 30^\circ$. Height h is opposite and d is adjacent, so use tangent in degree mode.

$$\tan \theta = \frac{h}{d}$$

$$h = d \tan \theta = 12 \tan 30^\circ \approx 6.93 \text{ m}$$

This is the height above the observer's horizontal sight level, not automatically the height above the ground.

Continue with practice sheet **1.5**. Solve before opening the solutions, and check both the method and the final units.

Exponential and logarithmic functions

These two functions describe growth and decay, and they undo each other.

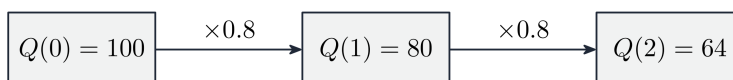
An **exponential function** has the form $y = a^x$ with $a > 0$ and $a \neq 1$: the variable is in the **exponent**. Each unit step multiplies its value by a . It grows when $a > 1$ and decays when $0 < a < 1$. Its y-intercept is 1 and its values stay positive; the x-axis is a horizontal asymptote.

A **logarithm** answers the reverse question. $\log_a y = x$ means exactly $a^x = y$, so $\log_{10} 1000 = 3$. For real logarithms the argument must be positive, and the base must be positive and different from 1. The graph $y = \log_a x$ passes through $(1, 0)$ and has vertical asymptote $x = 0$.

$$\log(mn) = \log m + \log n, \quad \log\left(\frac{m}{n}\right) = \log m - \log n, \quad \log(m^k) = k \log m$$

- For positive arguments and the same valid base, the laws turn multiplication into addition, which is what makes a logarithm useful for solving an equation with the unknown in the exponent.
- **Natural logarithms** use the base $e \approx 2.718$ and are written $\ln$.
- **Compound interest** is exponential: $A = P(1 + r)^n$ after n periods.
- **Exponential decay** has a base between 0 and 1, and can describe a retained proportion each period under stated assumptions. It does not guarantee that a real process follows that model indefinitely.

Worked example. repeated reductions multiply rather than subtract a fixed amount



equal time periods, changing absolute reductions

Known: an invented quantity starts at 100 units and decreases by 20% per whole period. The retained proportion is 0.8, so use a multiplicative model.

$$Q(n) = Q_0(1 - r)^n$$

$$Q(2) = 100(1 - 0.20)^2 = 64$$

The first decrease is 20 units and the second is 16. A fixed subtraction of 20 each period would be a different model.

Continue with practice sheet **1.6**. Solve before opening the solutions, and check both the method and the final units.