Exercise walk-through

4.5
Graphs and Networks

GAC Mathematics

Sets, relations and functions

- A **set** is a collection of distinct objects. Order and repetition do not matter.
- **Union** $A \cup B$ is everything in either; **intersection** $A \cap B$ is what is in both; the **set complement** is everything in the stated universe but outside the set.
- A **subset** has all its elements inside another set.

$$|F \cup G| = |F| + |G| - |F \cap G| = 18 + 15 - 7 = 26$$

Counting systems

- A **number base** b uses the digits 0 to $b-1$, and each place is worth a power of b. **Decimal** uses ten, **binary** two, **hexadecimal** sixteen (digits 0–9 then A–F).
- Every digit's value is its **place value**: in binary the places are 1, 2, 4, 8, 16 and so on.
- **Hexadecimal** is shorthand for binary: one hex digit is exactly four bits. Web colours use it: ‘#FF8000‘ means red 255, green 128, blue 0, a bright orange.

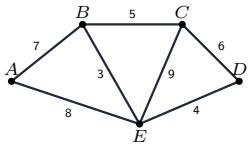
$$V = \sum d_i 2^i$$

Graphs and networks

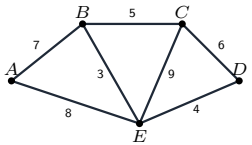
- A **graph** is a set of **vertices** joined by **edges**. It models anything with connections: roads, friendships, dependencies.
- The **degree** of a vertex is the number of **edges** that meet it. Every edge has two ends, so the degrees always add up to twice the number of edges. This is the fact behind Euler's Königsberg proof.
- A **tree** is a connected **graph** with no cycles; a **tree** with n vertices has $n - 1$ edges. A family tree and a computer's folder structure are trees.

Exercises 2.1·2.2

2.1 A simple undirected graph has edges AB, AC and BC. Find the degree of A and the total edge count. [2]

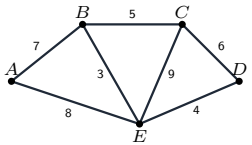


2.2 A tree has 9 vertices. Find its edge count and state the property excluding a triangle cycle. [2]



Exercises 2.1·2.2

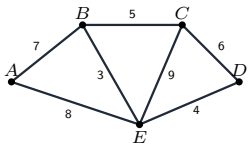
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2.1

A is incident to AB and AC, so degree 2. Total edge count 3.

2.2 A tree has 9 vertices. Find its edge count and state the property excluding a triangle cycle. [2]



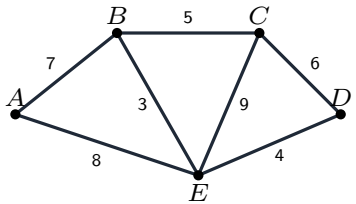
2.2

A tree has $n - 1 = 9 - 1 = 8$ edges. It is acyclic, so no triangle cycle is allowed.

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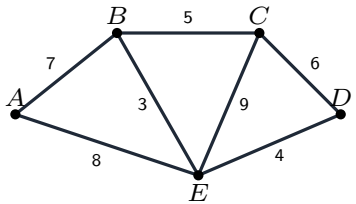
Exercise 2.3

2.3 A simple undirected graph has degrees 3, 2, 2, 1. Find its edge count and explain why degree sum 9 would be impossible. [3]



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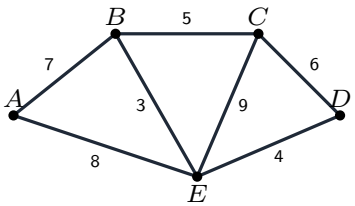
Each edge contributes two to total degree.

$$|E| = \frac{\sum d(v)}{2} = \frac{3+2+2+1}{2} = 4$$

Four edges. A sum of 9 is odd and cannot equal twice an integer edge count.

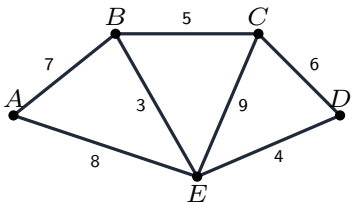
Exercise 2.4

2.4 A graph has edges AB, BC, CA and DE, on exactly five vertices A to E. It has n minus 1 edges. Decide whether it is a tree and justify your decision using two graph properties. [3]



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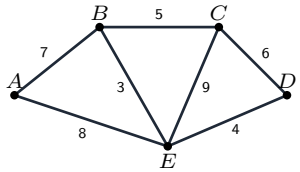


2.4

It is not a tree. The ABC triangle is a cycle. The DE component is disconnected from ABC. The four-edge count alone is insufficient.

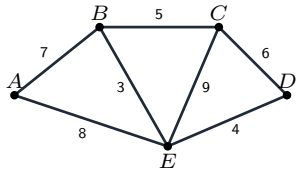
Exercise 3.1

3.1 A weighted undirected network has edges AB = 2, AC = 6, BC = 1, BD = 7, CD = 3. (a) Find a shortest path from A to D and its weight. (b) Find a minimum spanning tree and its total weight. (c) Explain the different requirements of these tasks. [6]



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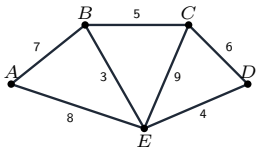
(a) A-B-C-D has weight $2 + 1 + 3 = 6$. Correct path earns B1 and weight B1. The other simple A-to-D alternatives are A-B-D with 9, A-C-D with 9, and A-C-B-D with 14, so 6 is shortest.

(b) Choose BC, AB and CD to connect all vertices without a cycle. Total weight 6.

(c) The shortest-path task connects only the specified endpoints with minimum route weight. The spanning-tree task connects all vertices with no cycles and minimum total included edge weight.

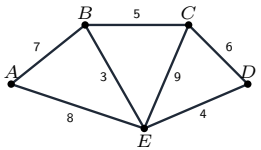
Exercise 3.2

3.2 A weighted undirected network on A, B, C, D and E has edges AB = 1, AC = 4, BC = 2, BD = 5, CD = 1, CE = 6, DE = 3. (a) Find a minimum spanning tree and its total weight, showing how cycles are avoided. (b) Find a shortest path from A to E and its weight. (c) If DE closes, find a new shortest A-to-E route using the remaining edges. Explain why that new route must be checked rather than retaining the old total. [8]



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3.2

(a) Select AB and CD, both weight 1. Add BC with weight 2, joining those components without a cycle. Add DE with weight 3 to connect E. Total $W = 1 + 1 + 2 + 3 = 7$, with all five vertices connected and four edges.

(b) Shortest path A-B-C-D-E. Weight $W_p = 1 + 2 + 1 + 3 = 7$. Nonnegative edge weights and comparison with alternative approaches to C/D establish no shorter route.

(c) E is now accessible only through CE, so the shortest remaining route is A-B-C-E with weight $1 + 2 + 6 = 9$. Correct route and weight. The closed edge cannot be used; the old path is infeasible and its total is not a valid remaining-network answer.

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