Exercise walk-through

3.3
Advanced Applications

GAC Mathematics

Differentiation

- The **power rule** : if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$.
- The **product rule** : $(uv)' = u'v + uv'$.
- The **quotient rule** , where $v \neq 0$: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation (2)

- The **chain rule** : if $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x)$.
- **Stationary points** occur where $f'(x) = 0$. The **second derivative** then classifies it: negative gives a local **maximum** , positive a local **minimum** . If the **second derivative** is zero, check the sign of $f'(x)$ on each side; $y = x^3$ at $x = 0$ is a stationary point that is neither.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

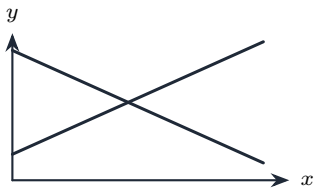
Integration

- An **indefinite integral** has no limits and needs the **constant of integration** : $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** has limits and gives a number: $\int_a^b f(x) \, dx = F(b) - F(a)$.
- **Integration by substitution** reverses the **chain rule**.
- △ Forgetting $+c$ on an **indefinite integral** loses marks on answers that are otherwise correct.

$$I = \int_{-1}^{1} x \, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

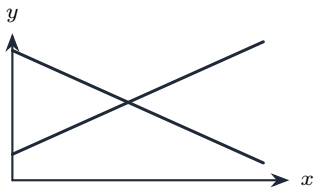
Exercise 2.1

2.1 A continuous model has cost $C(q) = 100 + 3q + 0.2q^2$. Find its marginal cost at q equal to 10. [2]



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2.1

Differentiate the cost model.

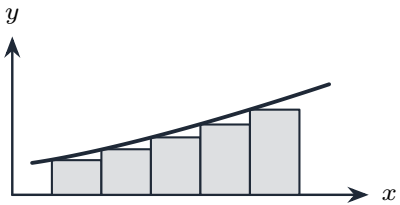
$$C'(q) = 3 + 0.4q$$

$$C'(10) = 3 + 0.4(10) = 7$$

Marginal cost 7 cost units per quantity unit.

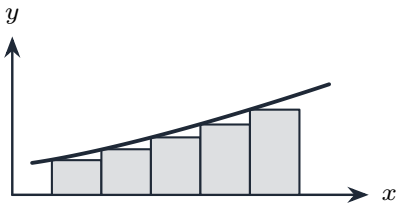
Exercise 2.2

2.2 A constant flow adds water at $r(t) = 6$ litres per minute for 5 minutes. Find the increase in volume. [2]



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2.2

Integrate the rate over the stated time.

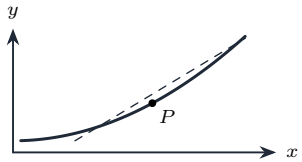
$$\Delta V = \int_0^5 r(t)dt = \int_0^5 6\,dt = 30 \text{ litres}$$

Increase 30 litres.

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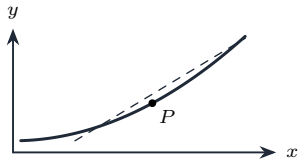
Exercise 2.3

2.3 A circle radius grows at $dr/dt = 0.5$ cm per second. Find its area growth rate when r equals 4 cm. [3]



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2.3

Use $A = \pi r^2$ with the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$$

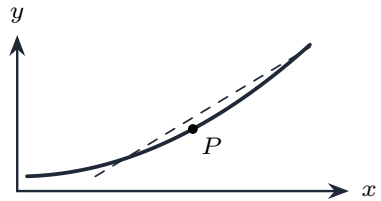
Correct related-rate relation.

$$\frac{dA}{dt} = 2\pi(4)(0.5) = 4\pi \ \frac{\text{cm}^2}{\text{s}}$$

Correct rate.

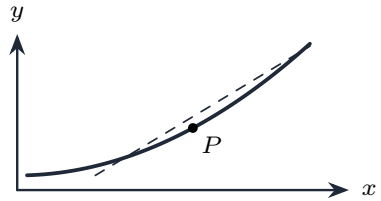
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2.4 For $C(q) = 100 + 3q + 0.2q^2$, compare $C(11) - C(10)$ with $C'(10)$. Explain the difference. [3]



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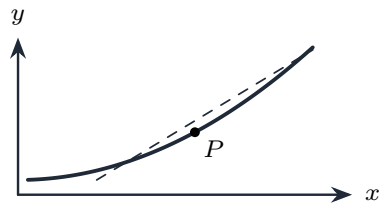
Compute the actual one-unit change.

$$\Delta C = C(11) - C(10) = 157.2 - 150 = 7.2$$

Correct finite increase 7.2. The derivative at 10 is 7, an instantaneous rate; the derivative changes over the next unit, so it need not equal the finite increase.

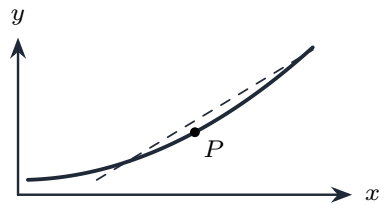
Exercise 3.1

3.1 A rectangle has perimeter 40 m. (a) Express area in terms of width x, state its feasible domain, and find the width maximising area. (b) Justify the maximum and find the maximum area. [6]



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3.1

The length is $20 - x$, with $0 < x < 20$.

$$A(x) = x(20 - x)$$

Correct area model. Differentiate and set to zero.

Full solution on next slide

Full solution 3.1

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Correct area model. Differentiate and set to zero.

$$A'(x) = 20 - 2x = 0 \quad \Longrightarrow \quad x = 10$$

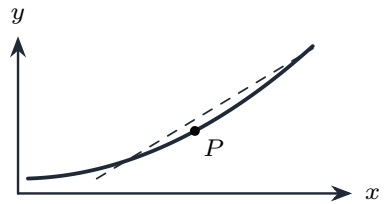
Width 10 m. The second derivative is negative 2 and area approaches zero at the endpoints, so this is the global maximum on the feasible domain.

$$A_{max} = 10(20 - 10) = 100 \text{ m}^2$$

Maximum area.

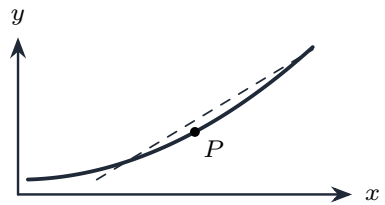
Exercise 3.2

3.2 A continuous teaching profit model is $P(q) = -q^2 + 11q - 10$ for $0 \leq q \leq 10$. (a) Find and justify its continuous maximum. (b) If q must instead be a whole number, find all quantities giving the largest feasible profit. (c) Find the break-even quantities and the whole-number quantities giving strictly positive profit. [8]



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3.2

(a) Differentiate.

$$P'(q) = -2q + 11 = 0 \quad \Longrightarrow \quad q = 5.5$$

Stationary quantity. Since $P''(q) = -2 < 0$ and both boundary profits are smaller, it is

Full solution on next slide

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Full solution 3.2 (1/2)

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$$P(5.5) = -(5.5)^2 + 11(5.5) - 10 = 20.25$$

Full solution 3.2 (2/2)

Continuous maximum profit.

(b) Check the adjacent integers 5 and 6.

$$P(5) = -25 + 55 - 10 = 20$$

$$P(6) = -36 + 66 - 10 = 20$$

Both quantities maximise feasible integer profit.

(c) Factor the zero-profit equation: $P(q) = -(q-1)(q-10) = 0$, giving break-even quantities 1 and 10. Strictly positive profit is $1 < q < 10$, hence whole numbers 2 through 9.