Exercise walk-through

3.2 Integration

GAC Mathematics

Differentiation

- The **power rule** : if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$.
- The **product rule** : $(uv)' = u'v + uv'$.
- The **quotient rule** , where $v \neq 0$: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation (2)

- The **chain rule** : if $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x)$.
- **Stationary points** occur where $f'(x) = 0$. The **second derivative** then classifies it: negative gives a local **maximum** , positive a local **minimum** . If the **second derivative** is zero, check the sign of $f'(x)$ on each side; $y = x^3$ at $x = 0$ is a stationary point that is neither.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

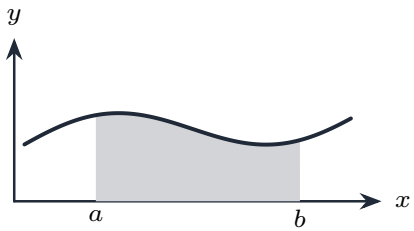
Integration

- An **indefinite integral** has no limits and needs the **constant of integration** : $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** has limits and gives a number: $\int_a^b f(x) \, dx = F(b) - F(a)$.
- **Integration by substitution** reverses the **chain rule**.
- △ Forgetting $+c$ on an **indefinite integral** loses marks on answers that are otherwise correct.

$$I = \int_{-1}^{1} x \, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

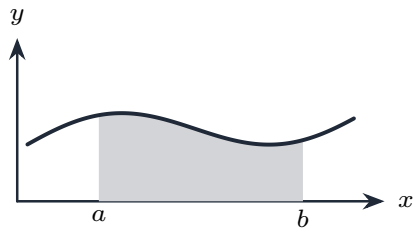
Exercise 2.1

2.1 Find $\int (3x^2 - 4)\,dx$. [2]



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2.1

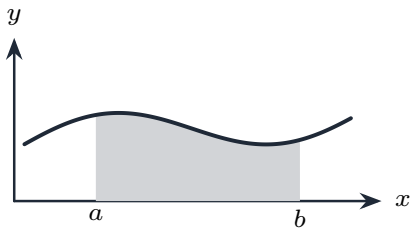
Use the power rule term by term.

$$F(x) = x^3 - 4x + C$$

Correct antiderivative including C.

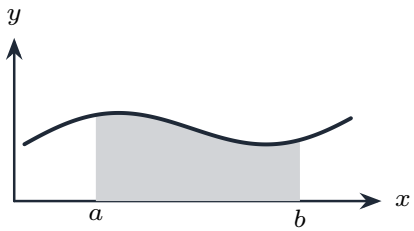
Exercise 2.2

2.2 Evaluate $\int_0^2 (2x+1)\,dx$. [2]



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2.2

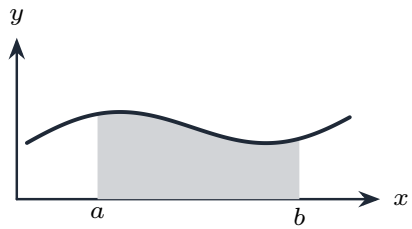
Use antiderivative $F(x) = x^2 + x$ and subtract the lower-bound value.

$$I = [x^2 + x]_0^2 = (2^2 + 2) - (0^2 + 0) = 6$$

Value 6.

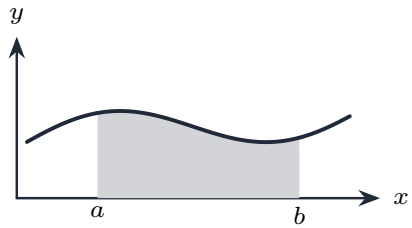
Exercise 2.3

2.3 Find $\int 6x/(3x^2+2)\,dx$ using substitution. [3]



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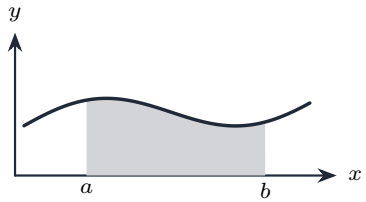
Let $u = 3x^2 + 2$, so $du = 6x\,dx$.

$$I = \int \frac{1}{u}\,du = \ln|u| + C$$

Logarithmic antiderivative. Since $3x^2 + 2 > 0$ for real x, $I = \ln(3x^2+2) + C$.

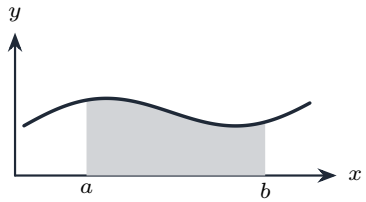
Exercise 2.4

2.4 Find both the signed integral $\int_{-1}^{1} x\,dx$ and the geometric area between that line and the x-axis over the same interval. [3]



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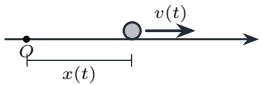
The signed integral is $[x^2/2]_{-1}^{1} = 0$. Split the geometric area at zero and reverse the sign of the lower region.

$$A = -\int_{-1}^{0} x\,dx + \int_{0}^{1} x\,dx = 1$$

Area 1.

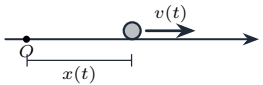
Exercise 3.1

3.1 A particle has velocity $v(t) = 4 - t$ metres per second for $0 \le t \le 6$ seconds and starts at position 5 m. Find (a) its displacement, (b) its total distance travelled, and (c) its final position. [6]



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3.1

(a) Integrate velocity for signed displacement.

$$\Delta s = \int_0^6 v(t)\,dt = \left[4t - \frac{t^2}{2}\right]_0^6 = 6 \text{ m}$$

Displacement 6 m.

(b) Velocity changes sign at $t = 4$ s. Split there and add magnitudes.

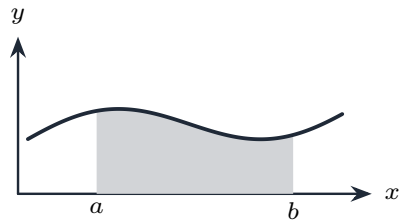
$$D = \int_0^4 (4 - t)\,dt - \int_4^6 (4 - t)\,dt = 8 + 2 = 10 \text{ m}$$

Distance 10 m.

(c) Final position $s_f = s_0 + \Delta s = 5 + 6 = 11$ m.

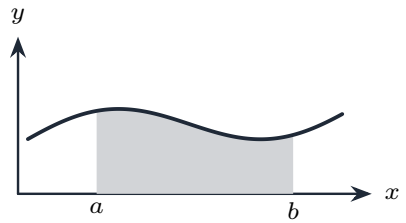
Exercise 3.2

3.2 Find the geometric area enclosed by $y = x$ and $y = x^2 - 2x$. (a) Find their intersection x-values. (b) Determine which curve is upper between them. (c) Form and evaluate an integral for the enclosed area. (d) Check the result by splitting your integral at the midpoint of the interval. [8]



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3.2

(a) Equate the curve values.

$$x = x^2 - 2x \quad \Longrightarrow \quad x(x-3) = 0$$

Full solution on next slide

Full solution 3.2

3.2

(a) Equate the curve values.

$$x = x^2 - 2x \quad \Longrightarrow \quad x(x-3) = 0$$

Intersections at x equal to 0 and 3.

(b) The difference is $x - (x^2 - 2x) = x(3-x) > 0$ for $0 < x < 3$, so the line is upper.

(c) Integrate upper minus lower.

$$A = \int_0^3 [x - (x^2 - 2x)]dx = \int_0^3 (3x - x^2)dx$$

$$A = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = \frac{9}{2}$$

Correct antiderivative. Area 9/2 square units.

(d) Each half has area $\int_0^{3/2}(3x - x^2)dx = 9/4$ and $\int_{3/2}^3(3x - x^2)dx = 9/4$. Correct half integrals. Their sum is 9/2, matching the result.