Exercise walk-through

1.6 Exponential and Logarithmic Functions

GAC Mathematics

Algebra: graphs and coordinate geometry

- Lines are **parallel** when their gradients are equal, and **perpendicular** when their gradients multiply to -1.
- The **midpoint** of a segment is the average of the endpoints; the **distance** between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graph is a **parabola**. Its **vertex** is the turning point; its **roots** are where it meets the x-axis. The path of a thrown ball is close to a parabola.
- To solve two equations **graphically**, draw both and read where they cross; check the answer algebraically, because a drawing is only approximate.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Trigonometry

- An angle can be measured in **degrees** or **radians**; π radians $= 180°$. Check your calculator's mode before every question.
- The **unit circle** extends sine and cosine beyond $90°$; they repeat every $360°$, so they are **periodic**.
- A **bearing** is measured clockwise from north and written with three figures, such as $075°$.
- **Angles of elevation and depression** are measured from the horizontal, up and down. Add the observer's eye height if the question asks for height above the ground.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Exponential and logarithmic functions

- The laws turn multiplication into addition, which solves equations with the unknown in the exponent.
- **Natural logarithms** use the base $e \approx 2.718$ and are written $\ln$.
- **Compound interest** is exponential: $A = P(1 + r)^n$.
- **Exponential decay** has a base between 0 and 1. Carbon-14 has a half-life of about 5,730 years, so after t years a fraction $\left(\frac{1}{2}\right)^{t/5730}$ remains; this is how archaeologists date wood and bone.

$$\log(mn) = \log m + \log n, \qquad \log\left(\frac{m}{n}\right) = \log m - \log n, \qquad \log(m^k) = k \log m$$

Exercise 2.1

2.1 Simplify $2^3 \times 2^{-5}$ to a single power and then to a fraction. [2]



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2.1

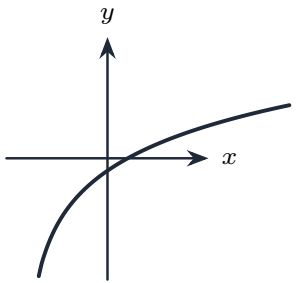
Same-base products add exponents.

$$E = 2^{3+(-5)} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

Correct power and fraction.

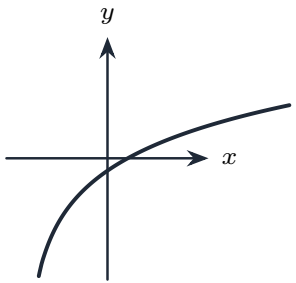
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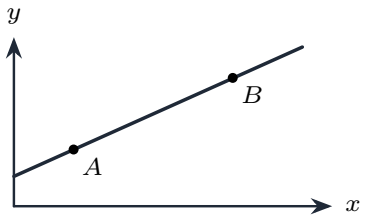
Convert the logarithmic statement to exponential form.

$$x = 3^4 = 81$$

Positive solution 81.

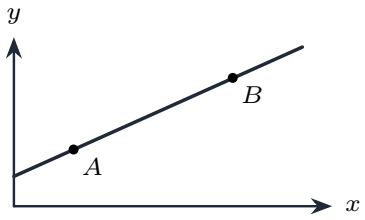
Exercise 2.3

2.3 Solve $5\times 2^t = 40$. Sketch $y = 2^t$, labelling its y-intercept and horizontal asymptote. [3]



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2.3

Divide both sides by 5.

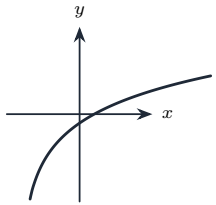
$$2^t = \frac{40}{5} = 8 = 2^3$$

Thus $t = 3$. An increasing exponential curve through $(0, 1)$, approaching but never meeting $y = 0$ as t decreases, earns the final point. Label the y-intercept and asymptote.

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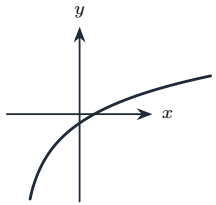
Exercise 2.4

2.4 Solve $\ln(x-1)+\ln 2=\ln 10$. State the domain restriction. Sketch $y=\ln x$, labelling its x-intercept and vertical asymptote. [3]



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2.4

The equation requires $x-1>0$, hence $x>1$. Combine logarithms of positive factors.

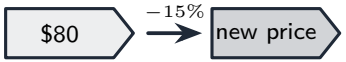
$$\ln(2(x-1))=\ln 10 \quad\Longrightarrow\quad 2(x-1)=10$$

$$x=1+\frac{10}{2}=6$$

Solution 6 satisfies $x>1$, and $\ln 5+\ln 2=\ln 10$. For the separate sketch, draw an increasing logarithmic curve on $x>0$ through $(1,0)$ with vertical asymptote $x=0$; label both features.

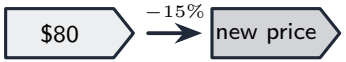
Exercise 3.1

3.1 An account starts with 500 units and grows by 4% at the end of each year, with no deposits, withdrawals or charges. (a) Write a model for the value after n whole years. (b) Find the value after 3 years to 2 decimal places. (c) Find the first whole year at which its value is greater than 600 units. [6]



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3.1

(a) Each year's multiplier is 1.04.

$$A(n) = 500(1.04)^n$$

Correct model.

Full solution on next slide

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Full solution 3.1

3.1

(a) Each year's multiplier is 1.04.

$$A(n) = 500(1.04)^n$$

Correct model.

(b) Substitute three completed periods.

$$A(3) = 500(1.04)^3 = 562.432 \approx 562.43$$

Correct rounded value.

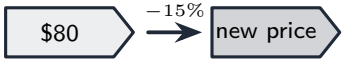
(c) The strict threshold is $500(1.04)^n > 600$. Divide by 500 and use the increasing natural logarithm.

$$n > \frac{\ln(600/500)}{\ln 1.04} \approx 4.6486$$

Correct threshold. First whole year is 5: $A(4) \approx 584.93 < 600$ and $A(5) \approx 608.33 > 600$.

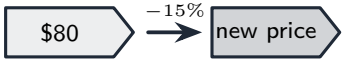
Exercise 3.2

3.2 A teaching model is $Q(t) = 200(0.85)^t$, where t is time in hours and may take any real value at least zero. (a) State the initial quantity and hourly percentage decrease. (b) Find Q(4) to 3 significant figures. (c) Find when the quantity first reaches 100 units, to 3 significant figures. (d) Under the same model sampled only after whole hours, find the first sampled value at or below 100. Explain whether the model ever reaches zero at a finite time. [8]



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(a) $Q(0) = 200$ units. Retained proportion is 0.85, so the hourly decrease is 15%.

(b) Substitute four hours.

$$Q(4) = 200(0.85)^4 = 104.40125 \approx 104 \text{ units}$$

Full solution on next slide

Full solution 3.2

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Correct 3 significant figures.

(c) Set the quantity equal to the target and take logarithms.

$$0.85^t = \frac{100}{200} = 0.5$$

$$t = \frac{\ln(100/200)}{\ln 0.85} \approx 4.27 \text{ hours}$$

Correct continuous time.

(d) First qualifying sample is at 5 hours because $Q(4) > 100$ and $Q(5) \approx 88.74 < 100$. For every finite t the positive base raised to t stays positive; the model approaches zero but never reaches it.