Exercise walk-through

1.3

Algebra: Graphs of Algebraic Relations and Coordinate Geometry

GAC Mathematics

Algebra I: introductory algebra

- To solve a **linear equation** , do the same operation to both sides until the letter stands alone.
- A **quadratic equation** has the form $ax^2 + bx + c = 0$ with $a \neq 0$. Factorise when you can, or use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** are two equations in two unknowns, solved by substitution or elimination. Two straight lines meet at one point, or never (parallel), or everywhere (the same line).
- △ An inequality **reverses** when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

$$(x+3)(x-5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

Algebra: graphs and coordinate geometry

- Lines are **parallel** when their gradients are equal, and **perpendicular** when their gradients multiply to -1.
- The **midpoint** of a segment is the average of the endpoints; the **distance** between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graph is a **parabola**. Its **vertex** is the turning point; its **roots** are where it meets the x-axis. The path of a thrown ball is close to a parabola.
- To solve two equations **graphically**, draw both and read where they cross; check the answer algebraically, because a drawing is only approximate.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

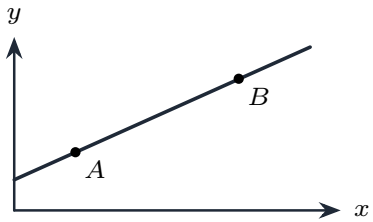
Trigonometry

- An angle can be measured in **degrees** or **radians**; π radians $= 180°$. Check your calculator's mode before every question.
- The **unit circle** extends sine and cosine beyond $90°$; they repeat every $360°$, so they are **periodic**.
- A **bearing** is measured clockwise from north and written with three figures, such as $075°$.
- **Angles of elevation and depression** are measured from the horizontal, up and down. Add the observer's eye height if the question asks for height above the ground.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

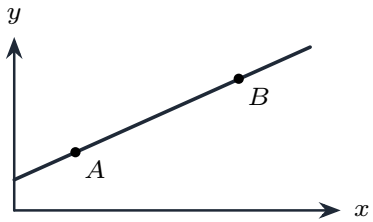
Exercise 2.1

2.1 Find the gradient of the line through $(2, 1)$ and $(6, 9)$. [2]



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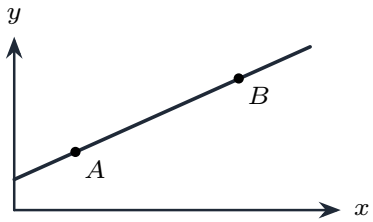
Use coordinate differences in the same order.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 1}{6 - 2} = 2$$

Gradient 2.

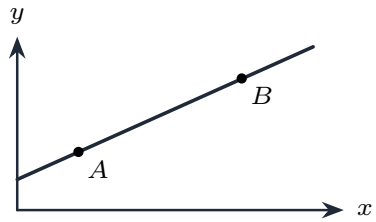
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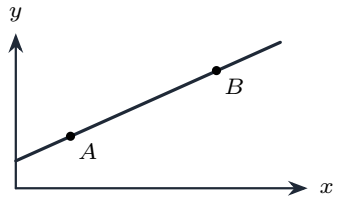
Average the corresponding coordinates.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{-2 + 4}{2}, \frac{5 - 1}{2} \right) = (1, 2)$$

Midpoint $(1, 2)$.

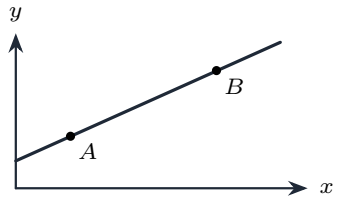
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Use $y = mx + c$ with the known point.

$$c = y - mx = 7 - 3(2) = 1$$

$$y = 3x + 1$$

Correct equation. Y-intercept 1, or point $(0, 1)$.

Exercise 2.4

2.4 For $y = (x-2)^2 - 9$, sketch the graph, labelling the vertex and both x-intercepts. Show how you find the intercepts. [3]

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Vertex form gives vertex $(2, -9)$. At an x-intercept y is zero.

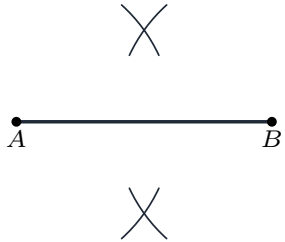
$$(x-2)^2 = 9 \quad \Longrightarrow \quad x - 2 = \pm 3$$

$$x = 2 \pm 3 = -1 \text{ or } 5$$

Intercepts $(-1, 0)$ and $(5, 0)$. An upward-opening parabola through these intercepts, with the labelled vertex below the axis, earns the final point.

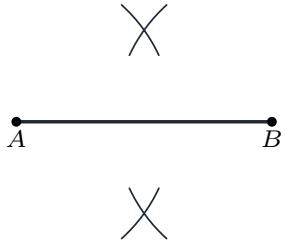
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3.1 Let $A = (1, 2)$ and $B = (7, 10)$. Find (a) the distance AB, (b) its midpoint, and (c) the equation of the perpendicular bisector of AB. [6]



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(a) Distance uses coordinate differences.

$$d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(7-1)^2 + (10-2)^2} = 10$$

Distance 10.

Full solution on next slide

Full solution 3.1 (1/2)

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(b) Average coordinates.

$$M = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right) = \left(\frac{1+7}{2}, \frac{2+10}{2}\right) = (4, 6)$$

Midpoint $(4, 6)$.

(c) The original gradient is $m_{AB} = (10-2)/(7-1) = 4/3$. Use its negative reciprocal.

$$m_{perp} = -\frac{1}{m_{AB}} = -\frac{3}{4}$$

The bisector passes through the midpoint.

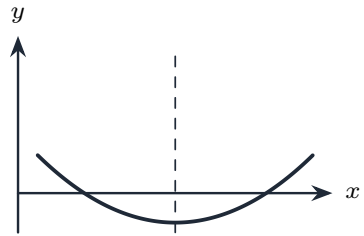
$$c = y_M - m_{perp} x_M = 6 - (-3/4)(4) = 9$$

$$y = -\frac{3}{4}x + 9$$

Correct equation.

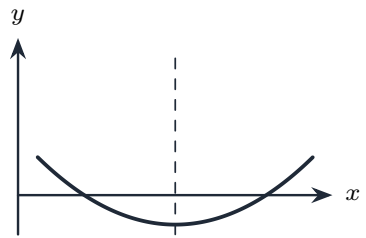
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3.2 A line has equation $y = x + 1$ and a parabola has equation $y = x^2 - 3x + 1$. (a) Find both intersections algebraically. (b) Find the parabola vertex. (c) Find the x-values for which the parabola lies strictly below the line. Justify the endpoints. [8]



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At an intersection both y-values agree.

$$x^2 - 3x + 1 = x + 1 \quad \Longrightarrow \quad x^2 - 4x = 0$$

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$$x^2 - 3x + 1 = x + 1 \quad \Longrightarrow \quad x^2 - 4x = 0$$

$$x(x - 4) = 0$$

Correct factorisation. Thus $x = 0$ or $x = 4$.

Full solution 3.2 (2/2)

$$y = x + 1 \quad \Longrightarrow \quad y_1 = 0 + 1 = 1, \quad y_2 = 4 + 1 = 5$$

Intersections $(0, 1)$ and $(4, 5)$.

Complete the square or use $x_v = -b/(2a)$.

$$y = (x - 3/2)^2 - 5/4$$

Vertex $(3/2, -5/4)$.

Below the line means $x(x - 4) < 0$, which holds between the roots: $0 < x < 4$. The endpoints give equal y-values, so strict inequality excludes them.