Vectors and transformation geometry

4MA1 full topic 5 package: Foundation 5.2 transformations and Higher 5.1 vectors with June 2024 Higher demand

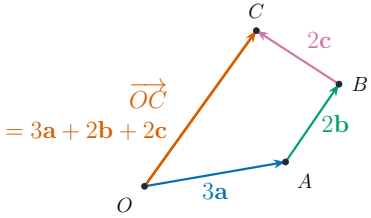
Vectors and transformation geometry

- write vectors in bold, as $\overrightarrow{AB}$ and as column vectors (Higher);
- add, subtract and multiply vectors, and find their **magnitude** (Higher);
- use vectors to prove facts about lines and shapes (Higher);
- carry out and fully describe reflections, rotations, translations and enlargements.

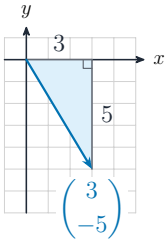
Vectors: notation to resultants

- $2\mathbf{p} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$, so $2\mathbf{p} + \mathbf{q} = \begin{pmatrix} 8-2 \\ -2+3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.
- **Magnitude** $= \sqrt{6^2 + 1^2} = \sqrt{37} = 6.08$ (3 s.f.).

Figure: Vectors: notation to resultants



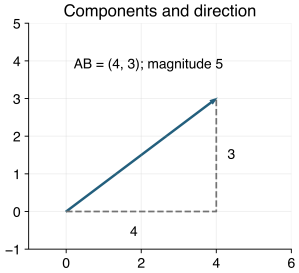
follow the route tip to tail;
the resultant joins start to end



modulus $= \sqrt{3^2 + 5^2}$
$= \sqrt{34} \approx 5.83$

- **Resultant** OC = 3a + 2b + 2c
- $|(3, -5)| = \sqrt{34} \approx 5.83$

Figure: Vectors: notation to resultants (2)



- AB = (4, 3) has **magnitude** $\sqrt{(4^2 + 3^2)} = 5$
- Match each part of the diagram to the text.

Vector geometry proofs

- To show two lines are **parallel**, show that one vector is a **scalar multiple** of the other.
- To show three points lie on one straight line, show that two of the vectors joining them are parallel and share a point.
- To find an unknown ratio, write the same vector in two different ways and compare the **coefficients 系数** of $\mathbf{a}$ and of $\mathbf{b}$.

Vector geometry proofs (2)

- In a **parallelogram** $\overrightarrow{AB} = \overrightarrow{OC} = \mathbf{c}$, so $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ and $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$.
- $\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = -\mathbf{a} + \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$.

Vector geometry proofs (3)

- The other **diagonal** is $\overrightarrow{AC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$, so $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$.
- So M is also the **midpoint** of AC: the diagonals cut each other in half.
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, so $\overrightarrow{OP} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

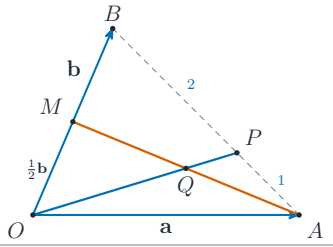
Vector geometry proofs (4)

- Q is on OP, so $\overrightarrow{OQ} = k\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$ for some number k.
- Q is also on AM, and $\overrightarrow{AM} = \frac{1}{2}\mathbf{b} - \mathbf{a}$, so $\overrightarrow{OQ} = \mathbf{a} + \lambda\left(\frac{1}{2}\mathbf{b} - \mathbf{a}\right) = (1-\lambda)\mathbf{a} + \frac{\lambda}{2}\mathbf{b}$ for some number λ.

Vector geometry proofs (5)

- Compare coefficients. $\mathbf{b}$: $\frac{1}{3}k = \frac{1}{2}\lambda$, so $\lambda = \frac{2}{3}k$. $\mathbf{a}$: $\frac{2}{3}k = 1 - \lambda = 1 - \frac{2}{3}k$, so $k = \frac{3}{4}$.
- So $OQ : OP = 3 : 4$. Since $\lambda = \frac{1}{2}$, Q is also the **midpoint** of AM.

Figure: Vector geometry proofs



write $\overrightarrow{OQ}$ along OP and along AM,
then equate: $k = \frac{3}{4}$, $\lambda = \frac{1}{2}$

- Write OQ two ways, along OP and along AM
- Equate coefficients: k = 3/4, so OQ : OP = 3 : 4

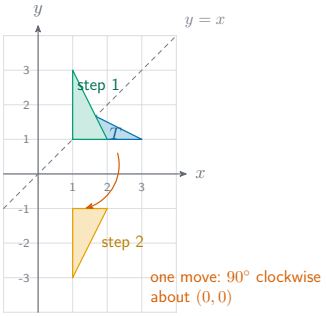
Transformations: rotation, reflection, translation

- A **reflection** 反射 flips the shape in a **mirror line** 镜像线, such as $x = 1$, $y = 2$ or $y = x$. Each point and its **image** are the same distance from the line.
- A **rotation** 旋转 turns the shape about a centre through an angle, **clockwise** 顺时针 or **anticlockwise** 逆时针.
- A **translation** 平移 slides the shape without turning it. Describe it with a **column vector**: $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ means 3 right and 2 down.

Transformations: rotation, reflection, translation (2)

- Follow a general point: $(x, y) \to (y, x) \to (y, -x)$.
- So $(1, 1) \to (1, -1)$, $(3, 1) \to (1, -3)$ and $(1, 2) \to (2, -1)$.
- The rule $(x, y) \to (y, -x)$ is a **rotation** of $90°$ **clockwise** about $(0, 0)$.

Figure: Transformations: rotation, reflection, translation



- Reflect in y = x, then in the x-axis
- Same as a 90° **clockwise rotation** about (0, 0)

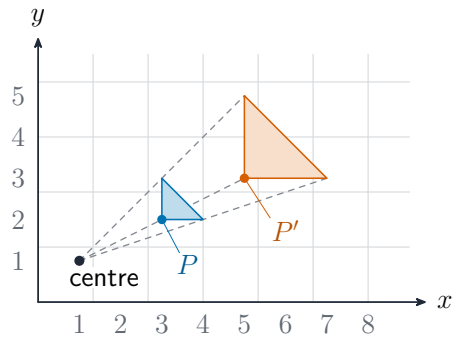
Enlargement and describing transformations

- From the centre to P is 2 across and 1 up.
- Double it: 4 across and 2 up from $(1, 1)$, so the **image** is $(5, 3)$.
- "**reflection** in the line $y = x$"

Enlargement and describing transformations (2)

- "**rotation** $90°$ **anticlockwise** about $(0, 1)$"
- "**translation** by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$"
- "enlargement, scale factor 2, centre $(1, 1)$".

Figure: Enlargement and describing transformations



- **Scale factor** 2 from (1, 1): P(3, 2) → (5, 3)
- The **image** is twice as far from the centre

Sources for this lesson

- Pearson Edexcel, *International GCSE (9–1) Mathematics A specification* (Issue 2, 2017), topic 5.
- Pearson Edexcel, 4MA1 Paper 2H, June 2024 (adapted).
- S. Stevin, *De Beghinselen der Weeghconst* (Leiden, 1586).

Common errors

- Writing a **column vector** as a coordinate, $(3, -5)$, instead of $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$.
- Forgetting to change the sign when going backwards along a vector.
- Assuming a point is a **midpoint** instead of finding the fraction.

Common errors (2)

- Describing a **rotation** without its centre or its direction.
- Reflecting in $y = x$ by swapping the signs instead of swapping the coordinates.
- Measuring an **enlargement** from the origin when the centre is somewhere else.
- Giving two transformations when the question asks for a single transformation.

Key terms

- **Vector**: a quantity with a size and a direction.
- **Resultant**: the single vector from the start to the end of a route.
- **Modulus**: the **magnitude**, or length, of a vector.

Key terms (2)

- **Image**: the shape after a transformation.
- **Centre of enlargement**: the fixed point that an **enlargement** is measured from.
- **Congruent**: exactly the same shape and size.
- **Similar**: the same shape, with all lengths multiplied by the same **scale factor**.

Check yourself

- 1.
- $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
- Find $3\mathbf{a} - \mathbf{b}$. 2.
- Find the **magnitude** of $\begin{pmatrix} -8 \\ 6 \end{pmatrix}$. 3.