Geometry and trigonometry

4MA1 full topic 4 package: Foundation 4.1-4.11 and Higher 4.6-4.11 with June 2024 Higher demand

Geometry and trigonometry

- use angle facts for lines, triangles, polygons and parallel lines, and give reasons;
- use bearings, **compound measures**, constructions, loci and scale drawings;
- use circle theorems (Higher) with a reason for every step;

Geometry and trigonometry (2)

- use **Pythagoras' theorem** and trigonometry in 2D and 3D, including the sine and cosine rules (Higher);
- find perimeters, areas, surface areas and volumes, and use similarity.

Angles, polygons and symmetry

- Angles on a straight line add up to $180°$; angles around a point add up to $360°$.
- Vertically opposite angles are equal.
- With parallel lines, **corresponding angles 同位角** are equal, **alternate angles 内错角** are equal, and co-interior (allied) angles add up to $180°$.

Angles, polygons and symmetry (2)

- The angles in a triangle add up to $180°$. The exterior angle of a triangle equals the sum of the two opposite interior angles.
- An isosceles triangle has two equal sides and two equal base angles. An equilateral triangle has three $60°$ angles.
- parallelogram: opposite sides parallel and equal, opposite angles equal, diagonals bisect each other

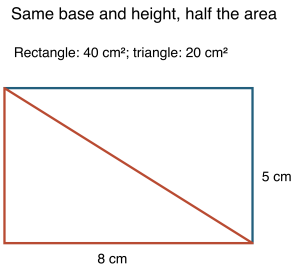
Angles, polygons and symmetry (3)

- rectangle: a parallelogram with four right angles
- rhombus: a parallelogram with four equal sides; its diagonals cross at right angles
- square: a rectangle and a rhombus at once

Angles, polygons and symmetry (4)

- kite: two pairs of equal adjacent sides
- trapezium: exactly one pair of parallel sides.

Figure: Angles, polygons and symmetry



- Rectangle $8 \times 5 = 40$ cm^2; the triangle with the same base and height is half: 20 cm^2
- Match each part of the diagram to the text.

Measures, bearings and compound rates

- **speed 速度** = distance ÷ time (km/h, m/s)
- **density 密度** = mass ÷ **volume** (g/cm^3)
- **pressure 压强** = force ÷ area (N/m^2).

Measures, bearings and compound rates (2)

- 36 minutes $= \frac{36}{60} = 0.6$ hours, so the time is 2.6 hours. Do not write 2.36.
- **Speed** $= 234 \div 2.6 = 90$ km/h.

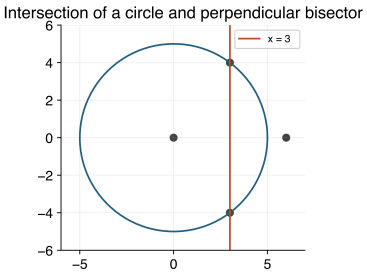
Constructions and scale drawings

- the perpendicular bisector of a line segment;
- the bisector of an angle;
- angles of $60°$ (from an equilateral triangle), $90°$ (a perpendicular), and $30°$ or $45°$ (by bisecting).

Constructions and scale drawings (2)

- a fixed distance from a point: a circle
- a fixed distance from a line: a pair of parallel lines
- the same distance from two points: the perpendicular bisector
- the same distance from two lines: the angle bisector.

Figure: Constructions and scale drawings



- Points 5 from O and equidistant from (0, 0) and (6, 0): $x = 3$ meets the circle at $(3, \pm 4)$
- Match each part of the diagram to the text.

Circle geometry with reasons

- The angle at the centre is twice the angle at the circumference on the same arc.
- The angle in a semicircle is $90°$.
- Angles in the same segment are equal.

Circle geometry with reasons (2)

- Opposite angles of a **cyclic quadrilateral** 圆内接四边形 (a **quadrilateral** with all four corners on a circle) add up to $180°$.
- The angle between a **tangent** and a **chord** equals the angle in the alternate segment.
- The perpendicular from the centre to a chord bisects the chord.

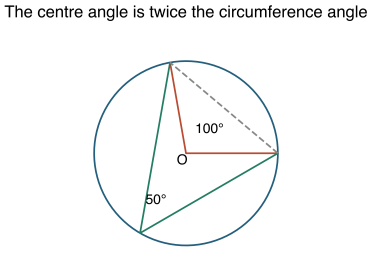
Circle geometry with reasons (3)

- Two tangents from an outside point are equal in length.
- If two chords AB and CD cross at E, then $AE \times EB = CE \times ED$.
- $\angle BAD = 180° - 128° = 52°$ (opposite angles of a **cyclic quadrilateral** add up to $180°$).

Circle geometry with reasons (4)

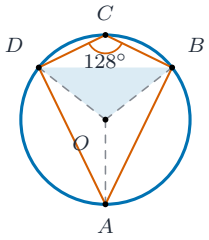
- $\angle BOD = 2 \times 52^\circ = 104^\circ$ (the angle at the centre is twice the angle at the circumference).
- $OB = OD$ (radii), so triangle OBD is isosceles and $\angle OBD = \frac{180^\circ - 104^\circ}{2} = 38^\circ$.

Figure: Circle geometry with reasons



- The angle at the centre is twice the angle at the circumference: 100° and 50°
- Match each part of the diagram to the text.

Figure: Circle geometry with reasons (2)



cyclic quad: $\angle BAD = 52°$
centre $= 2\times$ rim: $\angle BOD = 104°$
$OB = OD$ (radii): isosceles
$\angle OBD = \frac{180-104}{2} = 38°$

every numerical step names its reason

- Worked example 2: a circle centre O with **cyclic quadrilateral** ABCD, angle BCD = 128°, and the chain BAD = 52°, BOD = 104°, OBD = 38°.
- Match each part of the diagram to the text.

Pythagoras and trigonometry in 2D

- $\tan 62° = \frac{h}{50}$, so $h = 50 \tan 62° = 94.0$ m (3 s.f.).
- The **sine rule 正弦定理**, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, works when you know a side and its opposite angle.
- The **cosine rule 余弦定理**, $a^2 = b^2 + c^2 - 2bc\cos A$, works when you know two sides and the angle between them, or all three sides.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

Pythagoras and trigonometry in 2D (2)

- The area of a triangle is $\frac{1}{2}ab\sin C$. It also works backwards: if the area is 15, $a = 6$ and $b = 10$, then $30\sin C = 15$, so $\sin C = 0.5$ and $C = 30°$ (or $150°$ if the angle is obtuse).
- $BC = a$ is opposite A, and $AC = b$ is opposite B. By the **sine rule**, $\dfrac{b}{\sin 65°} = \dfrac{8}{\sin 40°}$.
- $b = \dfrac{8\sin 65°}{\sin 40°} = 11.3$ cm (3 s.f.).

Pythagoras and trigonometry in 2D (3)

- $a^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 110° = 130 + 43.1 = 173.1.$
- $a = 13.2$ cm (3 s.f.).
- $\cos 110°$ is negative, so the last term is added. The side opposite the obtuse angle is the longest side, which is a useful check.

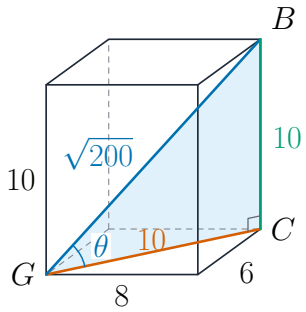
Trigonometry in three dimensions

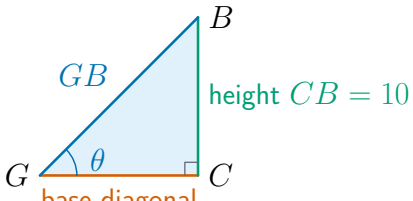
- Diagonal of the base: $\sqrt{6^2 + 8^2} = 10$ cm.
- **Space diagonal**: $\sqrt{10^2 + 10^2} = \sqrt{200} = 14.1$ cm.
- The angle θ with the base: $\tan\theta = \frac{10}{10} = 1$, so $\theta = 45°$.

Trigonometry in three dimensions (2)

- The angle is at M, so the side opposite it is BP.
- $\cos BMP = \dfrac{200 + 100 - 244}{2 \times \sqrt{200} \times 10} = \dfrac{56}{282.8} = 0.198.$
- $\angle BMP = 78.6°$ (1 d.p.).

Figure: Trigonometry in three dimensions





$GC = \sqrt{6^2 + 8^2} = 10$

find the right triangle GCB: $GB = \sqrt{200} \approx 14.1$, $\tan\theta = 1$, $\theta = 45°$

- Base diagonal 10, **space diagonal** $\sqrt{200}$
- $\tan\theta = 10/10$, so $\theta = 45°$

Mensuration: perimeter, area, surface and volume

- triangle: $\frac{1}{2}\times$ base $\times$ height
- parallelogram: base $\times$ height
- trapezium: $\frac{1}{2}(a+b)h$, where a and b are the parallel sides

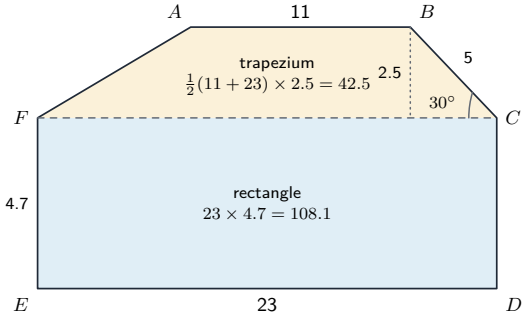
Mensuration: perimeter, area, surface and volume (2)

- circle: πr^2, with circumference $2\pi r$.
- Height of the trapezium: $5 \sin 30° = 2.5$ cm.
- Rectangle: $23 \times 4.7 = 108.1$ cm^2. Trapezium: $\frac{1}{2}(11 + 23) \times 2.5 = 42.5$ cm^2.

Mensuration: perimeter, area, surface and volume (3)

- Total area $= 150.6$ cm^2.
- Slant height: $l = \sqrt{3^2 + 4^2} = 5$ cm.
- $V = \frac{1}{3} \times \pi \times 9 \times 4 = 12\pi$ cm^3. Curved surface $= \pi \times 3 \times 5 = 15\pi$ cm^2.

Figure: Mensuration: perimeter, area, surface and volume



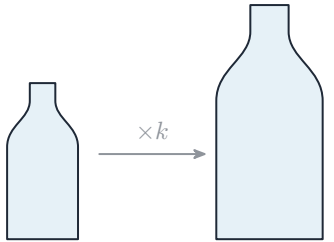
total area $108.1 + 42.5 = 150.6$ cm^2

- Rectangle 108.1 + trapezium 42.5 = 150.6 cm^2
- Trapezium height = 5 sin 30° = 2.5

Similarity: length, area and volume ratios

- Area factor: $k^2 = \frac{405}{180} = 2.25$, so the length factor is $k = 1.5$.
- **Volume** factor: $k^3 = 1.5^3 = 3.375$.
- Volume $= 320 \times 3.375 = 1080$ cm^3.
- $k^3 = \frac{125}{8}$, so $k = \sqrt[3]{\frac{125}{8}} = \frac{5}{2} = 2.5$.

Figure: Similarity: length, area and volume ratios



area: $k^2 = 405 \div 180 = 2.25$

length: $k = \sqrt{2.25} = 1.5$

volume: $k^3 = 1.5^3 = 3.375$

$320 \times 3.375 = 1080$ cm^3

- Area ratio, then length ratio, then **volume** ratio: 2.25, 1.5, 3.375. The larger bottle holds 1080 cubic centimetres, not 720
- Match each part of the diagram to the text.

Sources for this lesson

- Pearson Edexcel, *International GCSE (9–1) Mathematics A specification* (Issue 2, 2017), topic 4.
- Pearson Edexcel, 4MA1 Papers 1H and 2H, June 2024 (adapted).
- Cleomedes, *Caelestia* (Meteora), book 1, which records Eratosthenes' measurement.

Common errors

- Writing a **bearing** with two figures, or measuring it anticlockwise.
- Writing 2 hours 36 minutes as 2.36 hours.
- Giving an angle without a reason in a circle-theorem question.

Common errors (2)

- Using the wrong side as the hypotenuse; it is always opposite the right angle.
- Leaving the calculator in radians.
- Subtracting the last term of the **cosine rule** when the angle is obtuse; $\cos$ is negative, so it adds.

Common errors (3)

- Multiplying a **volume** by the area factor k^2 instead of the volume factor k^3.
- Forgetting the two radii in the perimeter of a **sector**.

Key terms

- **Regular polygon**: a polygon with all sides and all angles equal.
- **Bearing**: a three-figure angle measured clockwise from north.
- **Locus**: the set of points that obey a rule.

Key terms (2)

- **Cyclic quadrilateral**: a **quadrilateral** with all four corners on a circle.
- **Sine rule**: $\frac{a}{\sin A} = \frac{b}{\sin B}$, for any triangle.
- **Cosine rule**: $a^2 = b^2 + c^2 - 2bc\cos A$, for any triangle.
- **Scale factor**: the number every length is multiplied by in an enlargement.

Check yourself

- 1.
- Find the interior angle of a regular decagon (10 sides). 2.
- Q is on a **bearing** of $115°$ from P.
- Find the bearing of P from Q. 3.