Sequences, functions and graphs

4MA1 full topic 3 package: Foundation 3.1/3.3 and Higher 3.1-3.4 with June 2024 Higher demand

Sequences, functions and graphs

- find and use the nth term of a sequence, and the **sum of an arithmetic series**;
- use function notation, composite functions and inverse functions;
- find gradients and equations of straight lines, including **perpendicular lines**;

Sequences, functions and graphs (2)

- recognise and transform graphs, and find where graphs intersect;
- differentiate to find gradients, stationary points, velocity and acceleration.

Sequences and series

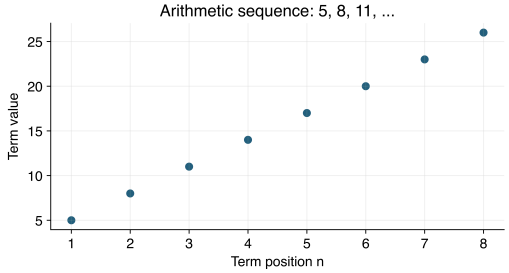
- $a + d = 7$ and $a + 4d = 19$. Subtract: $3d = 12$, so $d = 4$ and $a = 3$.
- $S_{50} = \frac{50}{2}\left[2 \times 3 + 49 \times 4\right] = 25 \times 202 = 5050$.
- The differences are equal: $(x+2) - (4x-14) = (7x-9) - (x+2)$, so $-3x + 16 = 6x - 11$ and $x = 3$.

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Sequences and series (2)

- The terms are -2, 5, 12, so $a = -2$ and $d = 7$.
- $S_{40} = 20\,[2(-2) + 39 \times 7] = 20 \times 269 = 5380$.

Figure: Sequences and series



- Add 3 each time: nth term $3n + 2$, so the eighth term is 26
- Match each part of the diagram to the text.

Function notation

- The **domain 定义域** is the set of inputs allowed. Leave out any input that does not work: for $f(x) = \frac{1}{x-2}$, $x \neq 2$; for $f(x) = \sqrt{x}$, $x \geq 0$.
- The **range 值域** is the set of outputs.
- The **composite function 复合函数** $\mathrm{fg}(x)$ means $f(g(x))$: apply g first, then f.

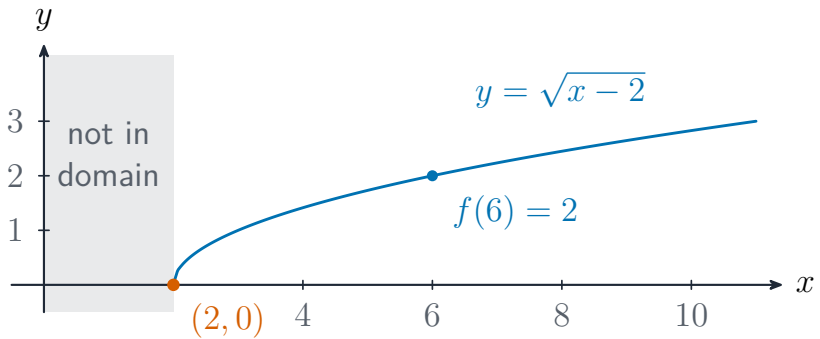
Function notation (2)

- The **inverse function 反函数** f^{-1} reverses f. To find it, write $y = f(x)$, rearrange to make x the subject, then swap x and y.
- $y = \sqrt{x-2}$, so $y^2 = x - 2$ and $x = y^2 + 2$.
- So $f^{-1}(x) = x^2 + 2$, for $x \geq 0$ (the **range** of f).

Function notation (3)

- Check: $f^{-1}(f(6)) = f^{-1}(2) = 6$.
- $g(k) = 3k + 1$, so $\mathrm{fg}(k) = \dfrac{3k+1}{2(3k+1)-4} = \dfrac{3k+1}{6k-2}$.
- $\dfrac{3k+1}{6k-2} = 2$, so $3k + 1 = 12k - 4$ and $k = \frac{5}{9}$.

Figure: Function notation

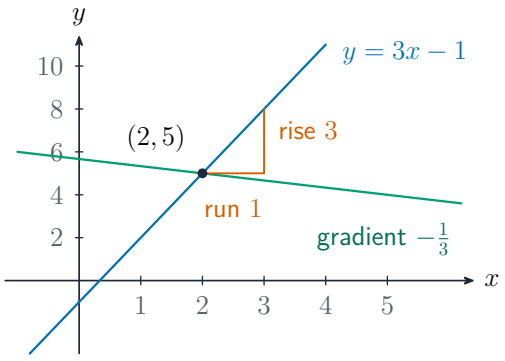


- **Domain** of $\sqrt{(x-2)}$: $x \geq 2$
- Find f^{-1}: rearrange, then swap x and y

Coordinates and straight lines

- $y = 3x + c$, and $5 = 3 \times 2 + c$, so $c = -1$ and $y = 3x - 1$.
- The **gradient** of AB is $\frac{5-2}{3-(-5)} = \frac{3}{8}$. BC is perpendicular to AB, so its gradient is $-\frac{8}{3}$.
- Through $B(3, 5)$: $y - 5 = -\frac{8}{3}(x - 3)$. Multiply by 3: $3y - 15 = -8x + 24$, so $8x + 3y - 39 = 0$.

Figure: Coordinates and straight lines



- **Gradient** = change in y ÷ change in x
- Perpendicular gradients multiply to -1

Graphs and transformations

- cubic graphs, $y = ax^3 + bx^2 + cx + d$: for example $y = x^3 - 4x$ rises, falls, then rises again
- reciprocal graphs, $y = \frac{a}{x}$: two separate curves that get closer to the axes but never touch them;
- $y = \sin x$ and $y = \cos x$, which repeat every $360°$ between -1 and 1, and $y = \tan x$, which repeats every $180°$;

Graphs and transformations (2)

- the circle $x^2 + y^2 = r^2$, with centre $(0, 0)$ and radius r.
- Substitute: $x^2 + (3x - 1)^2 = (3x - 1) + 11$, so $10x^2 - 6x + 1 = 3x + 10$ and $10x^2 - 9x - 9 = 0$.
- Factorise: $(5x + 3)(2x - 3) = 0$, so $x = -\frac{3}{5}$ or $x = \frac{3}{2}$.

Graphs and transformations (3)

- Using $y = 3x - 1$: the points are $\left(-\frac{3}{5}, -\frac{14}{5}\right)$ and $\left(\frac{3}{2}, \frac{7}{2}\right)$.
- Check $\left(\frac{3}{2}, \frac{7}{2}\right)$ in the curve: $\frac{9}{4} + \frac{49}{4} = 14.5$ and $\frac{7}{2} + 11 = 14.5$.

Differentiation and stationary points

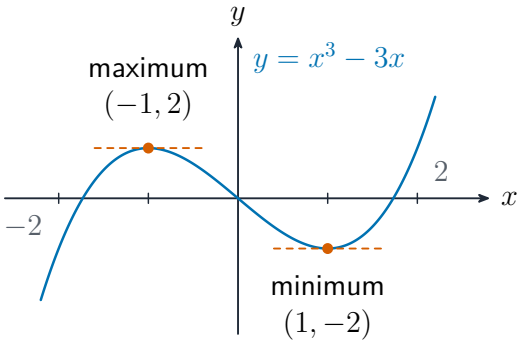
- $\frac{dy}{dx} = 3x^2 - 16x - 12 = (3x+2)(x-6) = 0$, so $x = -\frac{2}{3}$ or $x = 6$.
- At $x = 6$: $y = 216 - 288 - 72 + 5 = -139$. At $x = -\frac{2}{3}$: $y = \frac{247}{27} = 9.15$.
- The coefficient of x^3 is positive, so the curve rises, falls, then rises. The left point $\left(-\frac{2}{3}, \frac{247}{27}\right)$ is the **maximum**, and $(6, -139)$ is the **minimum**.

$$y = ax^n \implies \frac{dy}{dx} = anx^{n-1}$$

Differentiation and stationary points (2)

- $v = 3t^2 - 24$ m/s and $a = 6t$ m/s^2.
- At rest, $v = 0$: $t^2 = 8$, so $t = 2.83$ s (3 s.f.).

Figure: Differentiation and stationary points



- **Stationary point**: dy/dx = 0
- Kinematics: v = ds/dt, a = dv/dt

Sources for this lesson

- Pearson Edexcel, *International GCSE (9–1) Mathematics A specification* (Issue 2, 2017), topic 3.
- Pearson Edexcel, 4MA1 Papers 1H and 2H, June 2024 (adapted).
- R. Descartes, *La Géométrie* (Leiden, 1637).

Common errors

- Writing the nth term of 5, 8, 11, ... as $n + 3$; the **common difference** 3 must multiply n.
- Working out $\mathrm{fg}(x)$ by doing f first.
- Confusing $f^{-1}(x)$ with $\frac{1}{f(x)}$.

Common errors (2)

- Finding the **gradient** as change in x over change in y.
- Using the same gradient for **perpendicular lines**, or forgetting the minus sign in $-\frac{1}{m}$.
- Moving $y = f(x + 2)$ two units to the right; it moves to the left.
- Forgetting to substitute back to find the y-coordinate of a **stationary point**.

Key terms

- **Common difference**: the amount added each time in an **arithmetic sequence**.
- **Composite function**: a function made by applying one function and then another.
- **Inverse function**: the function that reverses another function.

Key terms (2)

- **Gradient**: the steepness of a line, the change in y divided by the change in x.
- **Differentiation**: finding the **gradient** function of a curve.
- **Stationary point**: a point on a curve where the gradient is zero.

Check yourself

- 1.
- Find the nth term of 7, 11, 15, 19, ... 2.
- $f(x) = 2x - 3$ and $g(x) = x^2$.
- Find $\mathrm{fg}(3)$ and $\mathrm{gf}(3)$. 3.