Equations, formulae and identities

4MA1 full topic 2 package: Foundation 2.1-2.4/2.6-2.8 and Higher 2.1-2.8 with June 2024 Higher demand

Equations, formulae and identities

- use **index laws** and manipulate algebraic expressions;
- change the subject of a formula, and use direct and **inverse proportion**;
- solve linear equations and simultaneous equations;

Equations, formulae and identities (2)

- factorise and solve quadratic equations, including by **completing the square**;
- solve inequalities, and pairs of linear and quadratic equations.

Symbols, indices and expressions

- multiply: add the **indices**, $x^3 \times x^4 = x^7$;
- divide: subtract the indices, $x^7 \div x^3 = x^4$;
- power of a power: multiply the indices, $(x^2)^5 = x^{10}$;

Symbols, indices and expressions (2)

- $x^0 = 1$ and $x^{-1} = \frac{1}{x}$; (Higher) $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{2}{3}} = \left(\sqrt[3]{x}\right)^2$, so $8^{\frac{2}{3}} = 2^2 = 4$.
- All the powers have **base** 3, so combine the **indices**: $(n-2) + (n+5) - (3n+10) = -n - 7$.
- $3^{-n-7} = 1 = 3^0$, so $-n - 7 = 0$ and $n = -7$.

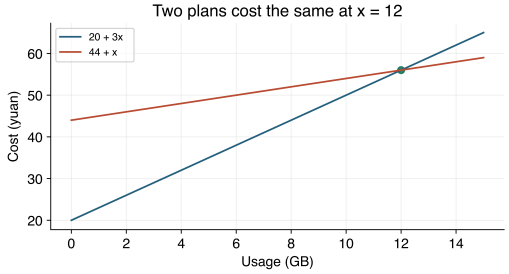
Symbols, indices and expressions (3)

- **substitution** : replace each letter by its value, for example $2a^2 - b$ with $a = -3$, $b = 4$ gives $18 - 4 = 14$;
- collecting **like terms** : $5x + 2y - 3x = 2x + 2y$;
- taking out a **common factor** : $6x^2 + 9x = 3x(2x + 3)$;

Symbols, indices and expressions (4)

- to **expand** brackets, multiply every term in one bracket by every term in the other: $(x+8)(x-5) = x^2 - 5x + 8x - 40 = x^2 + 3x - 40$.
- First two brackets: $(x+2)(x+3) = x^2 + 5x + 6$.
- Then $(x^2 + 5x + 6)(x-1) = x^3 - x^2 + 5x^2 - 5x + 6x - 6 = x^3 + 4x^2 + x - 6$.
- Check with $x = 0$: the left side is $2 \times 3 \times (-1) = -6$, the same as the **constant term** .

Figure: Symbols, indices and expressions



- The plans cost the same where $20 + 3x = 44 + x$: $x = 12$, cost 56 yuan
- Match each part of the diagram to the text.

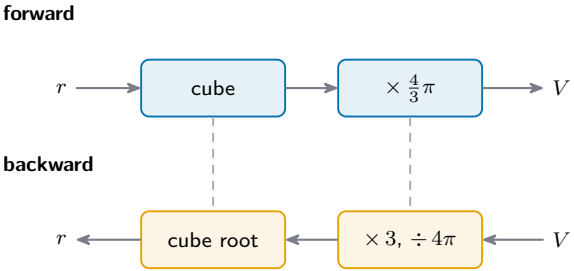
Formulae and changing the subject

- From $v = u + at$: subtract u, then divide by a: $t = \dfrac{v-u}{a}$.
- From $A = \pi r^2$: divide by π, then take the square **root**: $r = \sqrt{\dfrac{A}{\pi}}$.
- Divide by 2π: $\dfrac{T}{2\pi} = \sqrt{\dfrac{l}{g}}$.

Formulae and changing the subject (2)

- Square: $\dfrac{T^2}{4\pi^2} = \dfrac{l}{g}$.
- Multiply by g: $l = \dfrac{gT^2}{4\pi^2}$.
- $y = kx^3$, so $24 = 8k$ and $k = 3$.
- $y = 3x^3 = 3 \times 27 = 81$.

Figure: Formulae and changing the subject



- **Changing the subject** runs the machine backwards.
- r is the cube **root** of 3V over 4π.

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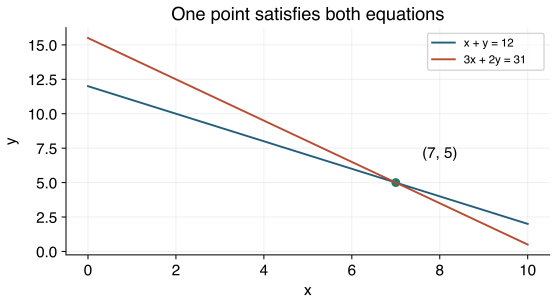
Linear equations, proportion and simultaneous pairs

- (a) $7x + 21 = 5x + 8$, so $2x = -13$ and $x = -6.5$.
- (b) Multiply every term by 15 to clear the fractions: $12x + 45 = 10$, so $12x = -35$ and $x = -\frac{35}{12}$.
- Multiply the first equation by 3 and the second by 2: $6x + 9y = 51$ and $6x - 10y = 70$.

Linear equations, proportion and simultaneous pairs (2)

- Subtract: $19y = -19$, so $y = -1$. Then $2x - 3 = 17$, so $x = 10$.
- Check in the second equation: $3(10) - 5(-1) = 35$.

Figure: Linear equations, proportion and simultaneous pairs



- The solution is the crossing point: x = 7, y = 5 satisfies both equations
- Match each part of the diagram to the text.

Quadratic algebra

- To factorise $x^2 + bx + c$, find two numbers that multiply to c and add to b: $x^2 + 10x + 24 = (x+6)(x+4)$.
- (Higher) To factorise $ax^2 + bx + c$, find two numbers that multiply to ac and add to b, then split the middle term: $6x^2 - 5x - 6 = 6x^2 - 9x + 4x - 6 = 3x(2x-3) + 2(2x-3) = (3x+2)(2x-3)$.
- The **difference of two squares** : $x^2 - a^2 = (x-a)(x+a)$, so $4x^2 - 9 = (2x-3)(2x+3)$.

$$\frac{x^2-4}{x^2+5x+6} = \frac{(x-2)(x+2)}{(x+2)(x+3)} = \frac{x-2}{x+3}$$

Quadratic algebra (2)

- Factorise: $3x^2 + x - 10 = (3x - 5)(x + 2)$.
- Dividing by a fraction means multiplying by its **reciprocal**:
  $(3x - 5) \times \dfrac{4x - 1}{(3x - 5)(x + 2)} = \dfrac{4x - 1}{x + 2}$.
- So the **expression** is $4 - \dfrac{4x - 1}{x + 2} = \dfrac{4(x + 2) - (4x - 1)}{x + 2} = \dfrac{9}{x + 2}$.

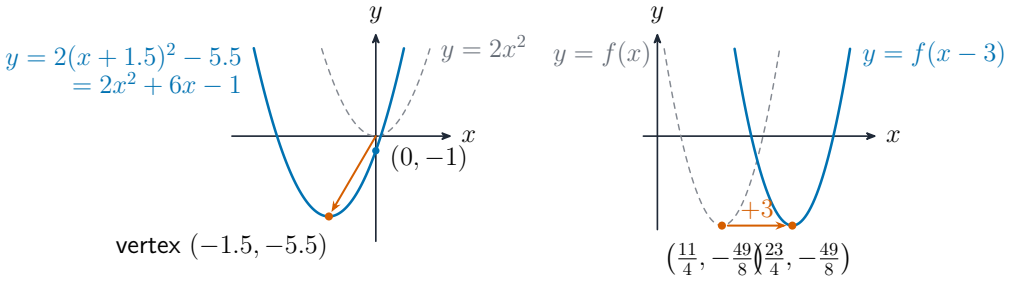
Completing the square and solving quadratics

- (a) Take out 2 from the x terms: $2\left(x^2 - \frac{11}{2}x\right) + 9$. Half of $\frac{11}{2}$ is $\frac{11}{4}$, so $2\left(x - \frac{11}{4}\right)^2 - 2 \times \frac{121}{16} + 9 = 2\left(x - \frac{11}{4}\right)^2 - \frac{49}{8}$.
- The minimum of f is at $\left(\frac{11}{4}, -\frac{49}{8}\right)$.
- (b) $y = f(x - 3)$ moves the graph 3 units to the right, so the minimum is at $\left(\frac{23}{4}, -\frac{49}{8}\right) = (5.75, -6.125)$.

Completing the square and solving quadratics (2)

- factorise: $x^2 + x - 30 = 0$ gives $(x+6)(x-5) = 0$, so the **roots** are $x = -6$ and $x = 5$;
- use the **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when it does not factorise;
- complete the square.
- $a = 2$, $b = 6$, $c = -1$: $x = \dfrac{-6 \pm \sqrt{36+8}}{4} = \dfrac{-6 \pm 2\sqrt{11}}{4} = \dfrac{-3 \pm \sqrt{11}}{2}$.

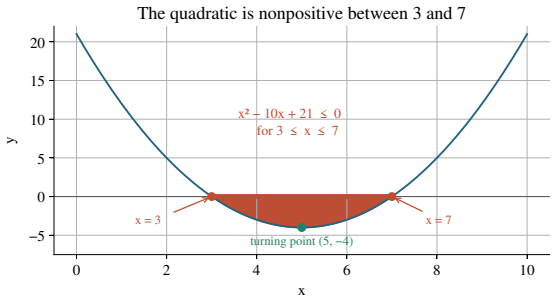
Figure: Completing the square and solving quadratics



$a(x+p)^2+q$ has its vertex at $(-p, q)$; $f(x-3)$ moves every point 3 to the right

- Left: **completing the square** shows the **vertex** of $y = 2x^2 + 6x - 1$ at $x = -1.5$. Right: worked example 9, the minimum moving 3 to the right under $f(x - 3)$.
- Match each part of the diagram to the text.

Figure: Completing the square and solving quadratics (2)



- $x^2 - 10x + 21 = (x - 3)(x - 7)$.
- It is ≤ 0 for $3 \leq x \leq 7$.

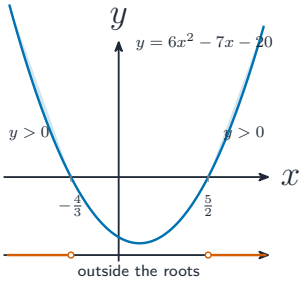
Inequalities and linear–quadratic pairs

- Factorise: $(3x+4)(2x-5) > 0$, so the critical values are $x = -\frac{4}{3}$ and $x = \frac{5}{2}$.
- The **parabola** opens upwards, so it is above the axis outside the roots: $x < -\frac{4}{3}$ or $x > \frac{5}{2}$.
- Substitute: $x^2 + (2x-11)^2 = 25$, so $5x^2 - 44x + 96 = 0$.

Inequalities and linear–quadratic pairs (2)

- $x = \dfrac{44 \pm \sqrt{1936 - 1920}}{10} = \dfrac{44 \pm 4}{10}$, so $x = 4.8$ or $x = 4$.
- The solutions are $(4.8, -1.4)$ and $(4, -3)$. On a graph, these are the two points where the line crosses the circle.

Figure: Inequalities and linear–quadratic pairs



positive quadratic: $>$ holds
outside the critical values

- Roots $-4/3$ and $5/2$.
- Positive outside the roots.

Sources for this lesson

- Pearson Edexcel, *International GCSE (9–1) Mathematics A specification* (Issue 2, 2017), topic 2.
- Pearson Edexcel, 4MA1 Papers 1H and 2H, June 2024 (adapted).
- R. Rashed, *Al-Khwarizmi: The Beginnings of Algebra* (Saqi Books, 2009).

Common errors

- Multiplying the bases when using **index laws**: $3^2 \times 3^4 = 3^6$, not 9^6.
- Expanding $(x+5)^2$ as x^2+25; it is $x^2+10x+25$.
- **Changing the subject** in the wrong order: undo the last operation first.

Common errors (2)

- Cancelling parts of terms: $\frac{x+2}{x+3}$ cannot be simplified.
- Forgetting the $\pm$ in the **quadratic formula**, and so losing a **root**.
- Forgetting to reverse the inequality sign when dividing by a negative number.
- Giving $-\frac{4}{3} < x < \frac{5}{2}$ for $6x^2 - 7x - 20 > 0$; sketch the **parabola** to see which side you need.

Key terms

- **Identity**: a statement that is true for every value of the **variable**, written with $\equiv$.
- **Changing the subject**: rearranging a formula so that a different letter stands alone.
- **Elimination**: solving simultaneous equations by adding or subtracting them to remove one letter.

Key terms (2)

- **Completing the square**: writing a quadratic in the form $a(x+p)^2+q$.
- **Critical value**: a value where a quadratic equals zero, which separates the ranges in an inequality.
- **Inverse proportion**: a relationship in which one quantity is proportional to the **reciprocal** of the other.

Check yourself

- 1.
- Simplify $\frac{x^5 \times x^3}{x^2}$. 2.
- **Expand** and simplify $(2x-3)(x+4)$. 3.
- Make x the subject of $y = \frac{x+5}{3}$. 4.