Further Pure Mathematics 2

Edexcel IAL FP2 (WFM02/01); optional pure unit beyond the P1–P4 + FP1 core

Further Pure Mathematics 2: inequalities, further complex numbers and differential equations

- solve inequalities involving the modulus sign, and sum series by the **method of differences**;
- use **Euler's relation** and **de Moivre's theorem**, and find the nth roots of complex numbers;
- sketch loci in the Argand diagram, and find images under transformations $w = \frac{az+b}{cz+d}$;

Further Pure Mathematics 2: inequalities, further complex numbers and differential equations (2)

- solve first and second order differential equations;
- find Maclaurin and **Taylor series**, and work with curves in **polar coordinates**.

Inequalities involving the modulus sign

- split into cases using the definition of the modulus, and keep only the solutions that lie in each case's own range;
- if both sides are moduli (so both are positive or zero), square both sides: $|A| > |B|$ means $A^2 > B^2$.
- Case $x^2 \geq 1$ (that is, $x \leq -1$ or $x \geq 1$): $x^2 - 1 > 2x + 2$, so $x^2 - 2x - 3 > 0$, $(x-3)(x+1) > 0$, giving $x < -1$ or $x > 3$. Both parts lie in this case's range.

Inequalities involving the modulus sign (2)

- Case $-1 < x < 1$: $1 - x^2 > 2x + 2$, so $x^2 + 2x + 1 < 0$, that is $(x+1)^2 < 0$, which is impossible.
- Solution: $x < -1$ or $x > 3$.
- For $x \geq 0$: $(x-2)(x+4) > 15x$ gives $x^2 - 13x - 8 > 0$, so $x > \dfrac{13 + \sqrt{201}}{2}$ (the other root is negative).

Inequalities involving the modulus sign (3)

- For $x < 0$: $(x-2)(4-x) > 15x$ gives $x^2 + 9x + 8 < 0$, so $-8 < x < -1$.
- For $x \geq 0$: $P = x^2 - 13x - 8$ and $Q = x^2 + 17x - 8$. Both are positive for $x > \dfrac{13 + \sqrt{201}}{2}$, and both are negative for $0 \leq x < \dfrac{\sqrt{321} - 17}{2}$.
- For $x < 0$: $Q = -x^2 + 21x - 8$ is always negative, so we need $P < 0$, which gives $x < -8$ or $-1 < x < 0$.
- Solution: $x < -8$, or $-1 < x < \dfrac{\sqrt{321} - 17}{2}$, or $x > \dfrac{13 + \sqrt{201}}{2}$.

The method of differences

- The **method of differences 差分法** sums a series by writing each term as a difference, usually using **partial fractions 部分分式**, so that most of the terms cancel.
- Write out the terms for $r = 1$, $r = 2$, ..., $r = n - 1$ and $r = n$ to show clearly what cancels; this earns marks.
- **Worked example 3.** Show that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$.
- Check with $n = 1$: both sides are $\frac{1}{6}$.

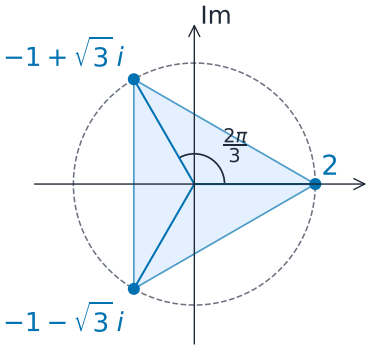
$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$$

Euler's relation and de Moivre's theorem

- **Euler's relation 欧拉关系式** is
- So a complex number with modulus r and argument θ can be written $z = re^{i\theta}$, and then $z^n = r^n e^{in\theta}$.
- It also gives $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.
- **De Moivre's theorem 棣莫弗定理** states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Figure: Euler's relation and de Moivre's theorem



$$z^3 = 8$$

$$z = 2\,e^{2k\pi i/3}$$

$$k = 0, 1, 2$$

- The three cube roots of 8 are equally spaced round a circle of radius 2.
- Match each part of the diagram to the text.

Loci, regions and z→w transformations

- $|z-a|=b$ is a circle with centre a and radius b;
- $|z-a|=|z-b|$ is the **perpendicular bisector 垂直平分线** of the line joining a and b;
- $|z-a|=k|z-b|$ with $k\neq 1$ is a circle: write $z=x+iy$, square both sides and complete the square;

Loci, regions and z→w transformations (2)

- $\arg(z-a)=\beta$ is a half-line starting at a.
- Make z the subject: $w+wz=9i-iz$, so $z(w+i)=9i-w$ and $z=\dfrac{9i-w}{w+i}$.
- $|z|=1$ gives $|9i-w|=|w+i|$: the **perpendicular bisector** of $9i$ and $-i$.
- With $w=u+iv$: $u^2+(v-9)^2=u^2+(v+1)^2$, so $81-18v=2v+1$ and $v=4$.

First order differential equations

- A first order linear differential equation has the form
- where P and Q are functions of x.
- Multiply every term by the **integrating factor 积分因子** $e^{\int P\,dx}$.
- The left side then becomes the derivative of a product, $\frac{d}{dx}\left(y\,e^{\int P\,dx}\right)$, so you can integrate both sides.

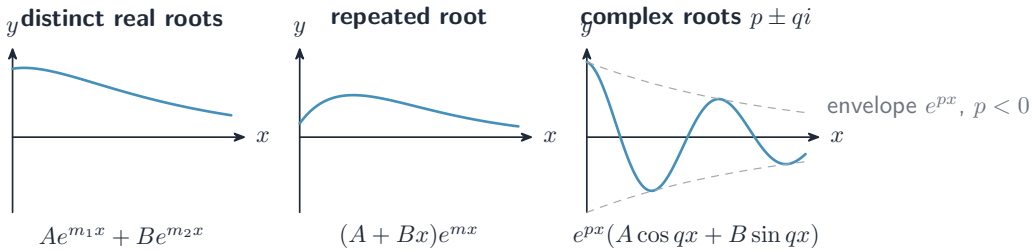
$$\frac{dy}{dx} + Py = Q$$

Second order differential equations

- The **complementary function** 余函数 (CF) solves the equation with $f(x) = 0$. Find it from the roots of the **auxiliary equation** 辅助方程 $am^2 + bm + c = 0$.
- A **particular integral** 特积分 (PI) is any one solution of the full equation. Try a function of the same form as $f(x)$.

$$y = 6e^{-x/4} - \frac{3}{2}e^{3x} + 6xe^{3x}$$

Figure: Second order differential equations



- The roots of the **auxiliary equation** set the shape of the **complementary function**; here every root has a negative real part.
- Match each part of the diagram to the text.

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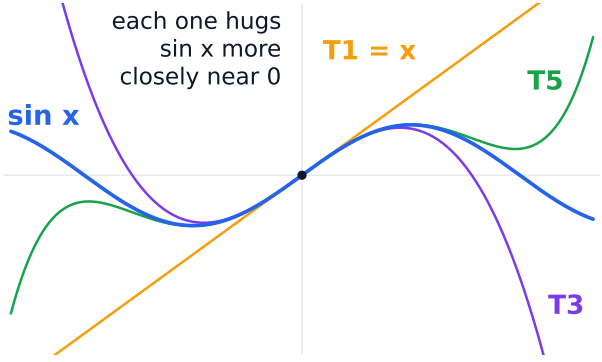
Maclaurin and Taylor series

- At $x = 3$: $y'' = 4y' + 5y + e^{2/3} = -4 + 10 + e^{2/3} = 6 + e^{2/3}$.
- Differentiate the equation: $y''' = 4y'' + 5y' + \frac{2}{9}e^{2x/9}$. At $x = 3$: $y''' = 24 + 4e^{2/3} - 5 + \frac{2}{9}e^{2/3} = 19 + \frac{38}{9}e^{2/3}$.
- So

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Figure: Maclaurin and Taylor series



- The curves compare $\sin x$ with $T_1 = x$, $T_3 = x - x^3/6$ and $T_5 = x - x^3/6 + x^5/120$.
- Read the axes and units first.

Polar coordinates and areas

- In **polar coordinates 极坐标**, a point is described by (r, θ): its distance r from the pole (the origin), and the angle θ measured anticlockwise from the **initial line 初始线** (the positive x-axis).
- To convert, use $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$.
- **Worked example 11.** Convert $r = 4\cos\theta$ to Cartesian form.
- Multiply by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$, which is the circle $(x-2)^2 + y^2 = 4$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (2018), unit FP2, and the formula booklet.
- L. Euler, *Introductio in analysin infinitorum*, volume 1 (Lausanne, 1748), chapter 8.
- Pearson Edexcel, WFM02/01 June 2025, questions 1 to 6, 8 and 9 (adapted).

Common errors

- Multiplying an inequality by an expression whose sign is unknown.
- Squaring an inequality when one side could be negative.
- Not writing out enough terms in the **method of differences** to show which terms are left.

Common errors (2)

- Giving only one nth root, or decimal arguments instead of exact ones.
- Forgetting to multiply Q by the **integrating factor** as well.
- Using λe^{px} as the **particular integral** when p is already a root of the **auxiliary equation**.

Common errors (3)

- Finding the constants from the **complementary function** alone instead of the complete solution.
- Using the wrong limits for one loop of a polar curve.

Key terms

- **Method of differences**: summing a series by writing each term as a difference so that most terms cancel.
- **Euler's relation**: $e^{i\theta} = \cos\theta + i\sin\theta$.
- **De Moivre's theorem**: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Key terms (2)

- **Integrating factor**: $e^{\int P\,dx}$, which turns the left side of $y' + Py = Q$ into the derivative of a product.
- **Complementary function**: the general solution of the equation with the right side set to zero.
- **Particular integral**: one solution of the complete differential equation.
- **Polar coordinates**: a distance from the pole and an angle from the **initial line**.

Check yourself

- 1.
- Solve $|2x-1| < |x+4|$. 2.
- Use the **method of differences** to find $\sum_{r=1}^{n} \frac{1}{r(r+1)}$. 3.
- Write $\left(\sqrt{3}+i\right)^6$ in the form $a+bi$. 4.