Pure Mathematics 4

Edexcel IAL P4 (WMA14/01); compulsory for IAL Pure Mathematics (and Mathematics)

Pure Mathematics 4: proof, parametrics and the full calculus

- prove results by contradiction;
- split **rational** functions into partial fractions;
- convert between parametric and **Cartesian** equations, and differentiate implicitly and parametrically;

Pure Mathematics 4: proof, parametrics and the full calculus (2)

- expand $(a + bx)^n$ for any **rational** n, and state when the expansion is valid;
- form and solve differential equations, and integrate by **substitution** and by parts;
- find volumes of revolution, and use vectors in three dimensions.

Proof by contradiction

- Squaring: $2 = \frac{p^2}{q^2} \implies p^2 = 2q^2$, so p^2 is even, so p is even: write $p = 2r$.
- Then $4r^2 = 2q^2 \implies q^2 = 2r^2$, so q is even too.
- Both p and q even contradicts $\frac{p}{q}$ being in lowest terms. ■

$$(\sin x+\cos x)^2 \geq 1 \implies \sin^2 x+2\sin x\cos x+\cos^2 x \geq 1 \implies 1+\sin 2x \geq 1 \implies \sin 2x \geq 0$$

Proof by contradiction (2)

- Consider $N = p_1 p_2 \cdots p_k + 1$. Dividing N by any p_i leaves remainder 1.
- So N is either prime itself or has a prime factor not in the list — either way a prime outside $\{p_1, \ldots, p_k\}$, contradicting the assumption. ■
- Suppose not: some x in the range has $|\sin x + \cos x| \geq 1$. Squaring (both sides non-negative):

Proof by contradiction (3)

- But $90° < x < 180°$ gives $180° < 2x < 360°$, where $\sin 2x < 0$ — a contradiction.
- So no such x exists and $|\sin x + \cos x| < 1$ throughout the range. ■
- Check at $x = 135°$: $|\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}| = 0 < 1$ ✓; at $x = 100°$: $|0.985 - 0.174| = 0.811 < 1$ ✓.

Partial fractions

- Check at $x = 0$: LHS $= \frac{1}{-2} = -\frac{1}{2}$; RHS $= \frac{3}{2} - 2 = -\frac{1}{2}$ ✓.
- Set up with the **repeated factor** split across both powers:
- Multiply through: $3x^2 + 7x + 6 = A(x+1)^2 + Bx(x+1) + Cx$.

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

Partial fractions (2)

- At $x = 0$: $6 = A$. At $x = -1$: $3 - 7 + 6 = 2 = -C \Rightarrow C = -2$.
- Compare x^2: $3 = A + B \Rightarrow B = -3$.
- Check at $x = 1$: LHS $= \frac{16}{4} = 4$; RHS $= 6 - \frac{3}{2} - \frac{1}{2} = 4$ ✓.

$$\frac{3x^2 + 7x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Figure: Partial fractions

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

multiply through: $5x + 1 = A(x-1) + B(x+2)$

put $x = 1$: $5(1) + 1 = B(1+2)$
$\Rightarrow B = 2$

put $x = -2$: $5(-2) + 1 = A(-2-1)$
$\Rightarrow A = 3$

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

check: $3(x-1) + 2(x+2) = 5x + 1$

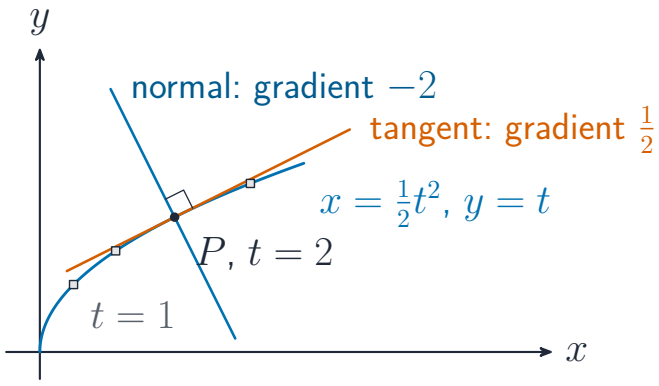
- Choose x to make one bracket zero: each **substitution** gives one constant
- Match each part of the diagram to the text.

Parametric equations

- From $x = 2t$: $t = \dfrac{x}{2}$, so
- Check at $t = 3$: $(x, y) = (6, 8)$; **Cartesian**: $36/4 - 1 = 8$ ✓.
- Normal gradient: $-\dfrac{1}{-4/5} = \dfrac{5}{4}$.
- Check: $5(-7) - 4(-3) + 23 = -35 + 12 + 23 = 0$ ✓.

$$y = \left(\frac{x}{2}\right)^2 - 1 = \frac{x^2}{4} - 1$$

Figure: Parametric equations



- Tangent: dy/dx = (dy/dt)/(dx/dt).
- Normal: the negative reciprocal gradient.

The binomial series for any rational n

- Factor: $\sqrt{4+x} = 2\left(1+\frac{x}{4}\right)^{1/2}$, valid $|x|<4$, with $n=\frac{1}{2}$:
- Check at $x=1$: series gives $2+0.25-0.0156=2.234$; true $\sqrt{5}=2.2361\ldots$ ✓ (the third term pulls the two-term value 2.25 down to within 0.002).

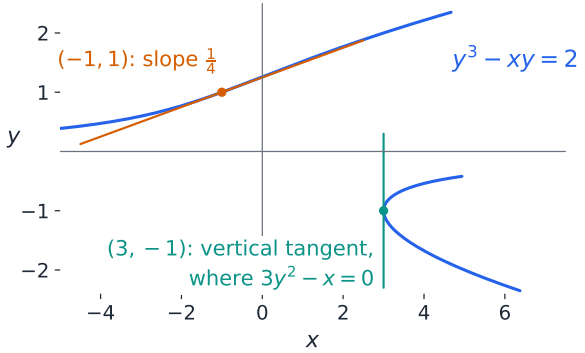
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Implicit and parametric differentiation

- Differentiate term by term:
- At $\left(\frac{1}{2}, 4\right)$: $\mathrm{e}^{1-1} = 1$, so
- Tangent: $y - 4 = \frac{38}{13}\left(x - \frac{1}{2}\right) \implies 38x - 13y + 33 = 0.$
- Check: $38\left(\frac{1}{2}\right) - 13(4) + 33 = 19 - 52 + 33 = 0$ ✓.

$$4y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} - 14\mathrm{e}^{1-2x} = 0 \implies \frac{dy}{dx} = \frac{6y + 14\mathrm{e}^{1-2x}}{4y - 6x}$$

Figure: Implicit and parametric differentiation



- Where the denominator of $\frac{dy}{dx}$ is zero (and the numerator is not), the tangent is vertical
- Read the axes and units first.

Differential equations and connected rates

- Differentiate the volume link: $\dfrac{dV}{dr} = 4\pi r^2$.
- Chain rule: $k = \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$, so
- The constant $\dfrac{k}{4\pi}$ multiplies $\dfrac{1}{r^2}$: the rate of decrease of the radius is inversely proportional to r^2. ■

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx$$

Differential equations and connected rates (2)

- Interpretation: small spheres vanish disproportionately fast — the last millimetre melts quickest.
- Separate: $\displaystyle\int y\,dy = \int 3\sec 2x\tan 2x\,dx$ (the right side integrates to $\frac{3}{2}\sec 2x$, reversing $\dfrac{d}{dx}\sec 2x = 2\sec 2x\tan 2x$):
- Substitute $\left(\frac{\pi}{6}, 4\right)$: $16 = 3\sec\frac{\pi}{3} + C = 3(2) + C$, so $C = 10$:
- Check: $3(2) + 10 = 16 = 4^2$ ✓.

$$\frac{dr}{dt} = \frac{k}{4\pi r^2} = \frac{k}{4\pi}\cdot\frac{1}{r^2}$$

Integration: substitution, parts and partial fractions

- **Substitution 换元积分** (usually given): rewrite the integral in u, adjust dx via $\frac{du}{dx}$, integrate, substitute back. It is the chain rule reversed.
- **Integration by parts 分部积分**: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ — the product rule reversed. Know $\int \ln x\,dx = x\ln x - x + c$ and repeated parts for $\int x^2 e^x dx$-type integrals.

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

Integration: substitution, parts and partial fractions (2)

- **Partial fractions** turn a **rational** integrand into sums of $\frac{A}{ax+b}$ (integrating to logs) and $\frac{C}{(ax+b)^2}$ (to powers).
- Parts with $u = e^x$, $\frac{dv}{dx} = \cos x$:
- Parts again on the new integral ($u = e^x$, $\frac{dv}{dx} = \sin x$):

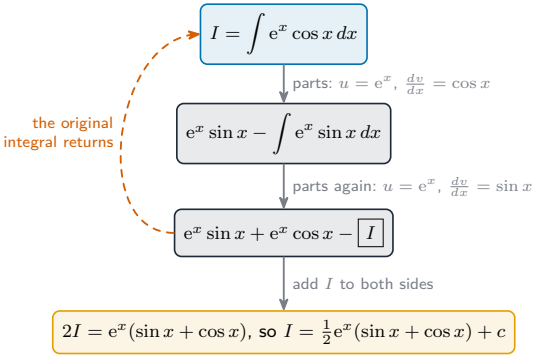
$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

Integration: substitution, parts and partial fractions (3)

- The original integral reappears; collect it:
- Check by differentiating: $\frac{1}{2}e^x(\sin x + \cos x) + \frac{1}{2}e^x(\cos x - \sin x) = e^x \cos x$ ✓.

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + c \implies \int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c$$

Figure: Integration: substitution, parts and partial fractions



- Two rounds of parts bring the original integral back.
- Collect it: 2I = e to the x (sin x + cos x).

Volumes of revolution and parametric areas

- Check: the average of $\cos^2 x$ over a half-turn is $\frac{1}{2}$, so $V \approx 9\pi \times \frac{1}{2} \times \frac{\pi}{2} = \frac{9\pi^2}{4}$ ✓ exactly.
- Rotating the region under $y = f(x) \geq 0$ about the x-axis sweeps a solid whose volume is

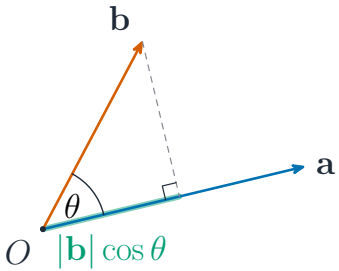
$$V = \pi \int_a^b y^2 \, dx$$

Vectors

- $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$, so $|\overrightarrow{AB}| = \sqrt{32}$; $\overrightarrow{AC} = \begin{pmatrix} 2 \\ \alpha - 2 \\ 8 \end{pmatrix}$.
- **Scalar product**: $\overrightarrow{AB} \cdot \overrightarrow{AC} = 8 + 0 + 32 = 40$.
- Angle condition with $\cos 45° = \frac{1}{\sqrt{2}}$:
- Check: $|\overrightarrow{AB}||\overrightarrow{AC}| \cos 45° = \sqrt{32} \times 10 \times \frac{1}{\sqrt{2}} = 4\sqrt{2} \times 10 \times \frac{1}{\sqrt{2}} = 40$ ✓ matches the scalar product.

$$\frac{40}{\sqrt{32}\sqrt{68 + (\alpha - 2)^2}} = \frac{1}{\sqrt{2}} \implies \sqrt{68 + (\alpha - 2)^2} = \frac{40\sqrt{2}}{\sqrt{32}} = \frac{40\sqrt{2}}{4\sqrt{2}} = 10$$

Figure: Vectors



- The **scalar product** picks out $|\mathbf{b}|\cos\theta$, how far $\mathbf{b}$ reaches along $\mathbf{a}$.
- Match each part of the diagram to the text.

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (Issue 3, 2019), unit P4.
- Euclid, *Elements*, Book IX, Proposition 20 (T. L. Heath's translation, Cambridge, 1908).
- Pearson Edexcel, WMA14/01 June 2025, questions 1 to 10 (adapted).

Common errors

- In a **proof by contradiction**, not stating the assumption clearly at the start, or not saying what the contradiction is.
- Using only one fraction for a **repeated factor**; $(x+1)^2$ needs $\frac{B}{x+1}$ and $\frac{C}{(x+1)^2}$.
- Expanding $(a+bx)^n$ without taking out a^n first.

Common errors (2)

- Forgetting the validity condition, such as $|x| < 4$ for $\sqrt{4+x}$.
- Differentiating y^2 as $2y$ instead of $2y\frac{dy}{dx}$.
- Forgetting the constant of integration before using the boundary condition.

Common errors (3)

- Forgetting to square y in $V = \pi \int y^2 \, dx$.
- Finding the angle between $\overrightarrow{BA}$ and $\overrightarrow{AC}$ instead of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Key terms

- **Proof by contradiction**: assuming that a statement is false and showing that this leads to something impossible.
- **Partial fractions**: simpler fractions that add up to a given **rational** function.
- **Parametric equations**: equations that give x and y in terms of a third variable, the parameter.

Key terms (2)

- **Particular solution**: the solution of a differential equation that fits a given condition.
- **Volume of revolution**: the volume of the solid formed by rotating a region about an axis.
- **Scalar product**: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$.

Check yourself

- 1.
- Write $\dfrac{7x-1}{(x-1)(x+2)}$ in partial fractions. 2.
- Find the **Cartesian** equation of the curve $x = t+1$, $y = t^2$. 3.
- Expand $(1-2x)^{-1}$ up to the term in x^2, and state when the expansion is valid. 4.