Course Companion

Pearson Edexcel · International A-Level · Pure Mathematics

Every topic handout in one volume

全部专题讲义合集

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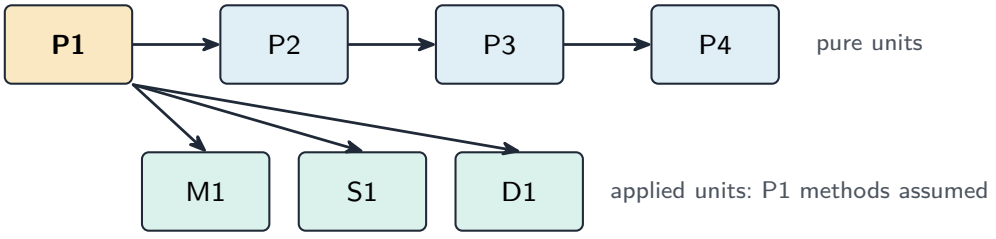
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1 · Pure Mathematics 1

Pearson Edexcel · International A-Level · Pure Mathematics

Pure Mathematics 1: the toolkit every later unit uses

Pure Mathematics 1 (P1) is the first compulsory unit of both International AS and International A Level Mathematics, and of International A Level Pure Mathematics. This reference covers every P1 topic in the Edexcel IAL specification (Issue 3, April 2019): algebra and functions, coordinate geometry, trigonometry, differentiation and integration.



P1 comes first: every later pure and applied unit assumes it

- The P1 paper (WMA11/01) lasts **1 hour 30 minutes** with **75 marks**, and you must answer all questions. Recent papers have ten questions from 5 to 11 marks each.
- A **calculator** 计算器 may be used, but not one with symbolic algebra, symbolic differentiation or integration, or retrievable stored formulae.
- The yellow booklet *Mathematical Formulae and Statistical Tables* is provided. The formulae in the "you must know" table below do **not** appear in that booklet, so memorise them.
- Give inexact answers to **3 significant figures** unless told otherwise. When a question says "you must show all stages of your working", a calculator answer alone scores nothing.

Formulae you must know for P1 (from the specification's assessment information):

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Formula	Use
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	roots of $ax^2 + bx + c = 0$
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	sine rule 正弦定理
$a^2 = b^2 + c^2 - 2bc\cos A$	cosine rule 余弦定理
area $= \frac{1}{2}ab\sin C$	area of a triangle
arc length $s = r\theta$, sector area $= \frac{1}{2}r^2\theta$	θ in radians 弧度
$\dfrac{d}{dx}x^n = nx^{n-1}$	power rule for differentiation
$\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$	power rule for integration

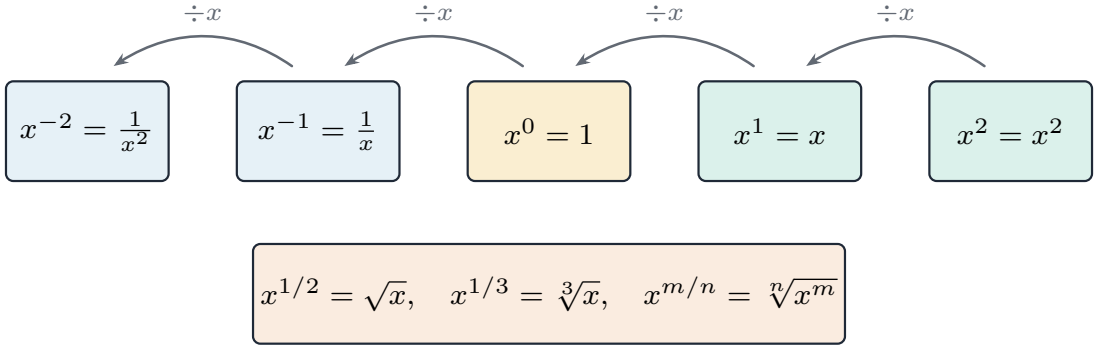
Algebra and functions

Indices and surds

For positive bases, the **laws of indices** 指数运算法则 hold for rational exponents. Zero and negative bases need their real-domain restrictions:

$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

Fractional powers are roots: $a^{m/n} = \sqrt[n]{a^m}$, so $a^{1/2} = \sqrt{a}$ and $a^{1/3} = \sqrt[3]{a}$. A negative power means reciprocal: $a^{-1} = 1/a$. Questions usually ask you to write an answer **in the form** kx^n with k and n fully simplified constants.



fractional powers are roots; negative powers are reciprocals

Negative powers are reciprocals; fractional powers are roots

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Worked example – three index manipulations (adapted from Edexcel IAL WMA11 June 2025 Q1). Given $p = \dfrac{x^4}{16}$ and $q = \dfrac{40}{3x}$, write each expression in the form kx^n.

(a) $p^{\frac{1}{2}}$.

- Known: $p = x^4/16$, and a power $\frac{1}{2}$ means square root.

$$p^{\frac{1}{2}} = \sqrt{\frac{x^4}{16}} = \frac{\sqrt{x^4}}{\sqrt{16}} = \frac{x^2}{4}$$

- Check: squaring the answer, $(x^2/4)^2 = x^4/16 = p$.

(b) $(pq)^{-1}$.

- Known: multiply before inverting, using $x^4 \cdot x^{-1} = x^3$.

$$pq = \frac{x^4}{16} \times \frac{40}{3x} = \frac{40x^3}{48} = \frac{5x^3}{6}$$

$$(pq)^{-1} = \left(\frac{5x^3}{6}\right)^{-1} = \frac{6}{5}x^{-3}$$

- Check: $(pq) \times (pq)^{-1} = \frac{5x^3}{6} \times \frac{6}{5x^3} = 1$.

(c) $(pq)^2$.

$$(pq)^2 = \left(\frac{5x^3}{6}\right)^2 = \frac{25x^6}{36}$$

- Check: $5^2 = 25$, $6^2 = 36$, $(x^3)^2 = x^6$, and the constants are already fully simplified.

A **surd** 根式 is an exact root such as $\sqrt{2}$. Keep answers exact unless asked for decimals; examiners penalise rounded decimals where a surd is expected. To **rationalise the denominator** 分母有理化, multiply numerator and denominator by a term that removes the root:

Worked example – rationalising. Write $\dfrac{12}{1+\sqrt{2}}$ with a rational denominator.

- Known: $(1+\sqrt{2})(\sqrt{2}-1) = \sqrt{2} - 1 + 2 - \sqrt{2} = 1$, so multiply by $\dfrac{\sqrt{2}-1}{\sqrt{2}-1}$.

$$\frac{12}{1+\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{12(\sqrt{2}-1)}{(1+\sqrt{2})(\sqrt{2}-1)} = 12(\sqrt{2}-1) = 12\sqrt{2} - 12$$

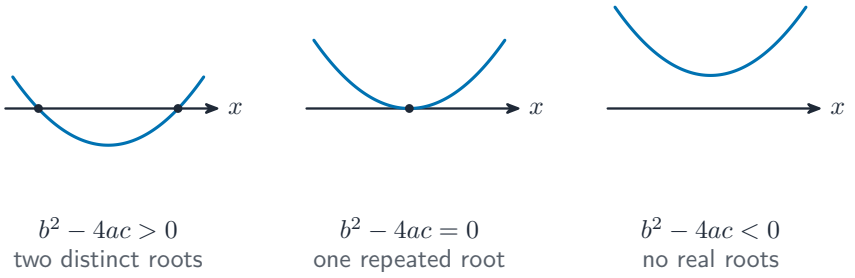
- Check by calculator: $12/(1 + 1.4142\ldots) = 4.9706\ldots$ and $12\sqrt{2} - 12 = 4.9706\ldots$ ✓

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■ Quadratics: discriminant and completing the square

A **quadratic function** 二次函数 has the form $y = ax^2 + bx + c$ with $a \neq 0$; its graph is a parabola opening up ($a > 0$) or down ($a < 0$). The **discriminant** 判别式 $\Delta = b^2 - 4ac$ counts the roots:

- $\Delta > 0$: two distinct real roots – the curve cuts the x-axis twice;
- $\Delta = 0$: one repeated root – the curve touches the x-axis;
- $\Delta < 0$: no real roots – the curve never meets the x-axis.



The discriminant counts intersections with the x-axis.

Completing the square 配方 rewrites ax^2+bx+c as $a\left(x+\dfrac{b}{2a}\right)^2+\left(c-\dfrac{b^2}{4a}\right)$, which gives the turning point $\left(-\dfrac{b}{2a},\, c-\dfrac{b^2}{4a}\right)$ directly. Solve quadratics by factorisation, the formula, completing the square, or a calculator – but "show all stages" questions reject calculator-only working.

> **Worked example – a hidden quadratic** (adapted from Edexcel IAL WMA11 June 2025 Q10). The equation $(k-1)x^6 + 4x^3 + (k-4) = 0$, where k is a constant.

(a) For $k = 4.5$, find the exact solutions.

- Known: with $k = 4.5$ the equation is $3.5x^6 + 4x^3 + 0.5 = 0$; multiply by 2 to clear decimals:
$$7x^6 + 8x^3 + 1 = 0$$
- Why substitute: $x^6 = (x^3)^2$, so this is a quadratic in x^3. Substitute $z = x^3$:
$$7z^2 + 8z + 1 = 0 \implies (7z+1)(z+1) = 0 \implies z = -\tfrac{1}{7} \text{ or } z = -1$$
- Return to x: $x^3 = -\frac{1}{7}$ or $x^3 = -1$, so
$$x = -\sqrt[3]{\tfrac{1}{7}} = -\frac{1}{\sqrt[3]{7}} = -\frac{\sqrt[3]{49}}{7} \qquad \text{or} \qquad x = -1$$
- Check: $z = -1$ gives $7 - 8 + 1 = 0$ ✓; $z = -\frac{1}{7}$ gives $7 \cdot \frac{1}{49} - \frac{8}{7} + 1 = \frac{1}{7} - \frac{8}{7} + 1 = 0$ ✓

(b) Find the set of values of k for which the equation has **no real roots**.

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- Known: treat the equation as a quadratic in x^3 with $a = k-1$, $b = 4$, $c = k-4$. No real roots needs a negative discriminant:

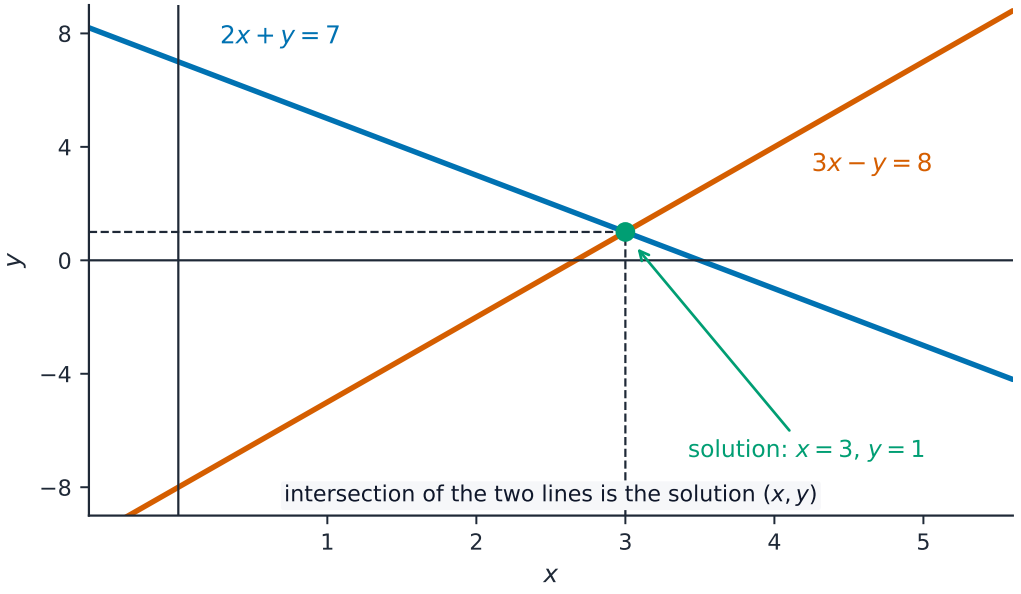
$$b^2 - 4ac = 16 - 4(k-1)(k-4) = 16 - 4(k^2 - 5k + 4) = -4k^2 + 20k = 4k(5-k)$$

$$4k(5-k) < 0 \implies k < 0 \text{ or } k > 5$$

- Check: at $k = -1$: $4(-1)(6) = -24 < 0$ ✓; at $k = 6$: $4(6)(-1) = -24 < 0$ ✓; at $k = 3$: $4(3)(2) = 24 > 0$ ✓ (two real roots there).

■ Simultaneous equations and inequalities

Solve **simultaneous equations** 联立方程 analytically by **substitution** 代入法: rearrange the linear equation, substitute into the other, solve, then substitute back. A line meeting a curve gives a quadratic whose roots are the intersection x-coordinates – this is the "geometrical interpretation of algebraic solutions" the specification asks for.



The solution of simultaneous equations is where their graphs cross

Worked example – line meets cubic-squared curve (adapted from Edexcel IAL WMA11 June 2025 Q7). C_1: $y = 0.2x^2(x-4)$ and C_2: $y = 1.2x(8-x)$ intersect where

$$0.2x^2(x-4) = 1.2x(8-x)$$

- Expand both sides: $0.2x^3 - 0.8x^2 = 9.6x - 1.2x^2$.
- Collect terms: $0.2x^3 + 0.4x^2 - 9.6x = 0$, then divide by 0.2: $x^3 + 2x^2 - 48x = 0$.
- Factorise: $x(x^2 + 2x - 48) = x(x+8)(x-6) = 0$, so $x = 0, -8, 6$.
- Substitute back into C_2: $(0, 0)$, $(-8, -153.6)$, $(6, 14.4)$.

- Check with C_1: $0.2(-8)^2(-12) = 0.2 \cdot 64 \cdot (-12) = -153.6$ ✓ and $0.2(36)(2) = 14.4$ ✓

Inequalities 不等式 follow the same algebra, but the answer is a range:

- Solve $ax+b > cx+d$ by collecting x (the inequality sign survives adding/subtracting, and survives dividing by a **positive** number; dividing by a negative flips the sign).
- Solve $px^2 + qx + r \leq 0$ by sketching: the solution is where the parabola is on or below the axis: between two roots when $p > 0$, and outside them when $p < 0$. Check repeated/no-real-root cases separately.
- "Curve below line" inequalities such as $px^2 + qx + r < ax + b$ mean: where is the curve's height less than the line's height? Find intersections first.
- When **sketching** the solution region, use a **dotted line** for strict $<$ or $>$ boundaries and a **solid line** for $\leq$ or $\geq$.

> **Worked example.** Solve $x^2 < 3x + 10$.

- Rearrange: $x^2 - 3x - 10 < 0$, and factorise: $(x - 5)(x + 2) < 0$.
- The parabola $y = (x - 5)(x + 2)$ opens upward, so it is below zero **between** its roots:

$$-2 < x < 5$$

- Check: $x = 0$ gives $0 < 10$ ✓; $x = 6$ gives $36 < 28$ × (correctly excluded).

■ Polynomials: expanding and factorising

Expand brackets and collect like terms; factorise by taking out common factors and by grouping. Cubics factorise fully once one root is spotted: $x^3 + 4x^2 + 3x = x(x^2 + 4x + 3) = x(x + 1)(x + 3)$. The notation $f(x)$ names the function, so $f(2)$ means "substitute $x = 2$".

■ Sketching graphs and transformations

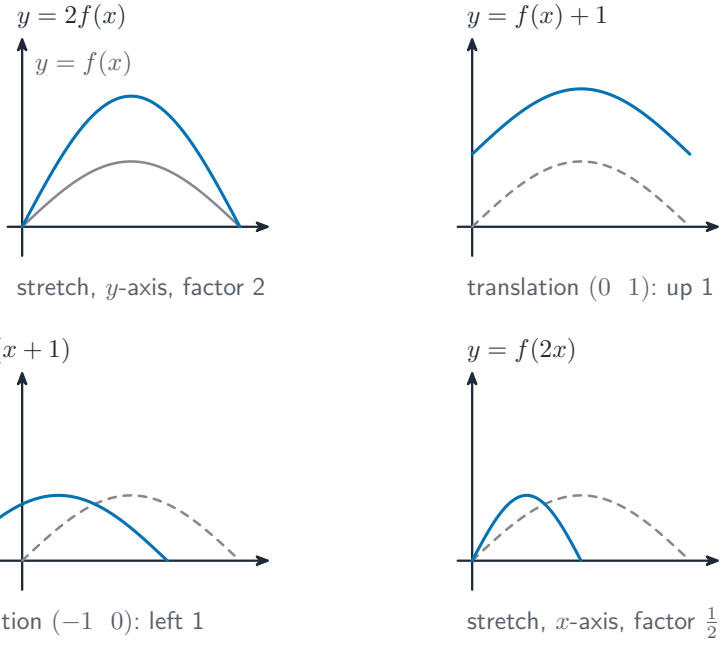
Sketch (do not plot) these families: quadratics and simple **cubics** 三次函数 such as $y = x^3 - 4x$; **reciprocal functions** 反比例函数 $y = \dfrac{k}{x}$ and $y = \dfrac{k}{x^2}$ $(x \neq 0)$, which have an **asymptote** 渐近线 – a line approached in the relevant limit; a horizontal or oblique asymptote may be crossed at a finite input; and the trigonometric graphs below. Mark intercepts and turning points.

Intersections answer equations: the curves $y = f(x)$ and $y = g(x)$ meet where $f(x) = g(x)$.

One **transformation** 变换 at a time, applied to $y = f(x)$:

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New equation	What it does to the graph
$y = af(x)$	stretch parallel to the y-axis, scale factor a
$y = f(x) + a$	translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$ (moves up by a)
$y = f(x + a)$	translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ (moves left by a)
$y = f(ax)$	stretch parallel to the x-axis, scale factor $\dfrac{1}{a}$



Name the direction, the type and the scale factor (or vector).

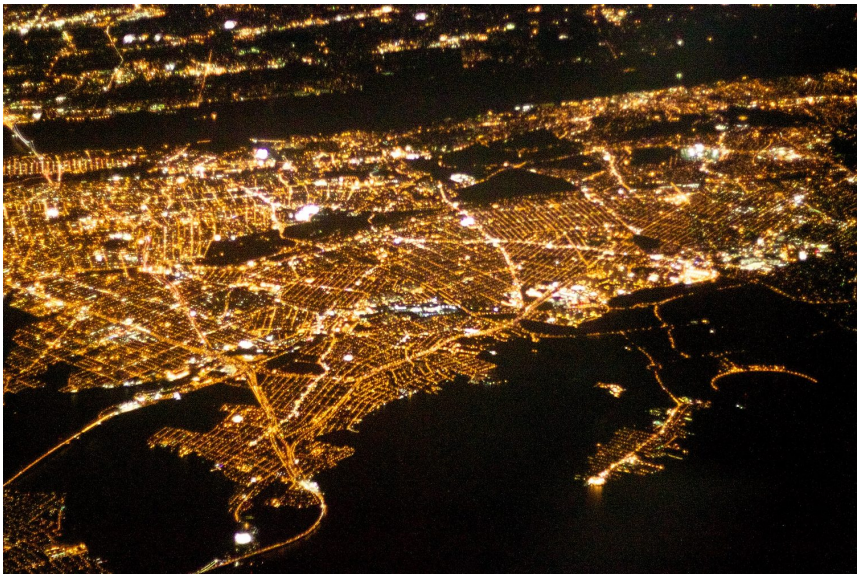
Worked example – describe two transformations (adapted from Edexcel IAL WMA11 June 2025 Q9).

- $y = \sqrt{2x}$ from $y = \sqrt{x}$: write $\sqrt{2x} = f(2x)$, so this is a **stretch parallel to the x-axis with scale factor $\frac{1}{2}$**.
- $y = 12 - \sqrt{x}$ from $y = -\sqrt{x}$: write $12 - \sqrt{x} = -\sqrt{x} + 12$, a **translation by** $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$ (up by 12).
- Check with a point: $y = \sqrt{x}$ passes $(4, 2)$; after the first transformation the curve passes $(2, 2)$ (halved x), and $y = \sqrt{2 \times 2} = 2$ ✓. After the second, $y = -\sqrt{4} + 12 = 10$, so $(4, 10)$ lies on it ✓.

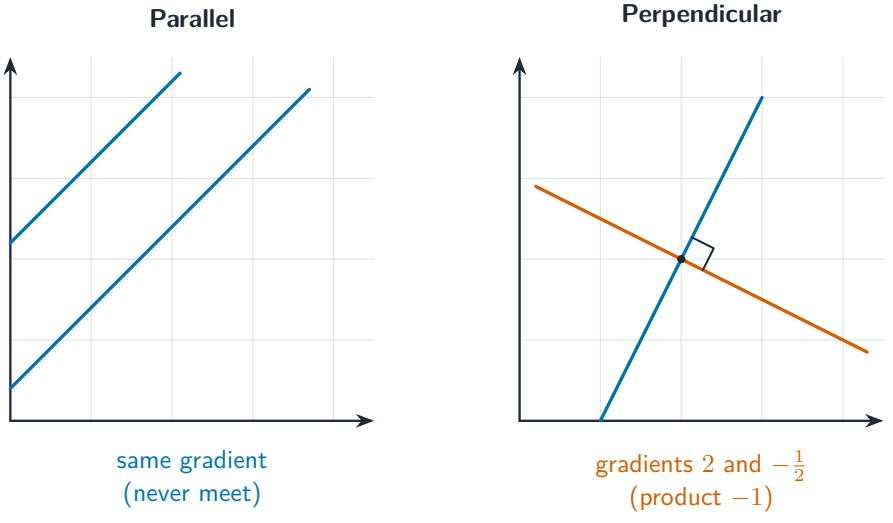
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Coordinate geometry in the (x, y) plane

A line through (x_1, y_1) with gradient m has equation $y - y_1 = m(x - x_1)$; the form $ax + by + c = 0$ hides the gradient in $-\dfrac{a}{b}$. Two points give the gradient: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



A city street grid: every point is fixed by two coordinates, and straight roads are straight lines



The non-vertical lines shown have equal gradients when parallel and gradient product -1 when perpendicular.

- Distinct non-vertical **parallel** 平行 lines share their gradient: $m_1 = m_2$; vertical parallel lines have undefined gradients.

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- **Perpendicular** 垂直 lines with finite gradients satisfy $m_1 m_2 = -1$. A horizontal line and a vertical line are also perpendicular.

> **Worked example – perpendicular through the origin** (adapted from Edexcel IAL WMA11 June 2025 Q5). The line l_1 has equation $x - 2y + 25 = 0$; l_2 passes through the origin and is perpendicular to l_1.

(a) Equation of l_2.

- Known: rearrange l_1: $2y = x + 25$, so $y = \frac{1}{2}x + \frac{25}{2}$ and $m_1 = \frac{1}{2}$.
- Why: perpendicular gradients multiply to -1, so $m_2 = -1 \div \frac{1}{2} = -2$.

$$y - 0 = -2(x - 0) \implies y = -2x$$

(b) l_1 and l_2 intersect at P.

- Substitute $y = -2x$ into $x - 2y + 25 = 0$:

$$x - 2(-2x) + 25 = 0 \implies 5x + 25 = 0 \implies x = -5, \quad y = -2(-5) = 10$$

- $P = (-5, 10)$. Check in l_1: $-5 - 20 + 25 = 0$ ✓

(c) Shortest distance from l_1 to the origin, as a simplified surd.

- The shortest distance is $|OP|$ along the perpendicular:

$$|OP| = \sqrt{(-5-0)^2 + (10-0)^2} = \sqrt{25 + 100} = \sqrt{125} = 5\sqrt{5}$$

- Check: $\sqrt{125} = \sqrt{25 \times 5} = 5\sqrt{5} \approx 11.18$ ✓

Trigonometry

■ The sine rule, cosine rule and area of a triangle

In triangle ABC with sides a, b, c opposite angles A, B, C:



A Ferris wheel: a seat's height rises and falls like a sine curve as the wheel turns

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$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad a^2=b^2+c^2-2bc\cos A \qquad \text{area}=\tfrac{1}{2}ab\sin C$$

Use the sine rule for pairs (side, opposite angle); the cosine rule when no pair is opposite. The sine rule has an **ambiguous case** 二义情形: $\sin\theta=\sin(180°-\theta)$, so an angle and its supplement can both fit – check which the diagram allows.

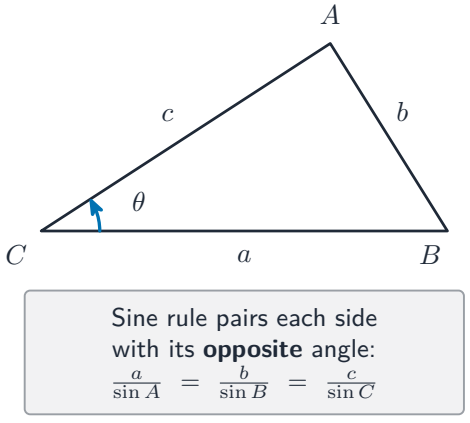
Radian measure, arc length and sector area

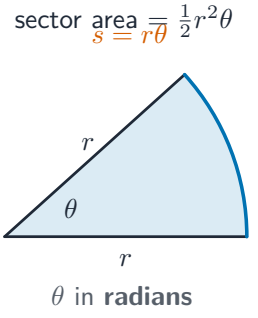
One **radian** is the angle subtended by an arc equal in length to the radius; $180°=\pi$ radians. With θ in radians:

$$\text{arc length } s=r\theta \qquad \text{sector area}=\tfrac{1}{2}r^2\theta$$

The **arc length** 弧长 is $s=r\theta$ and the **sector area** 扇形面积 is $\frac{1}{2}r^2\theta$.

A **major sector** is the larger share of the circle: angle $2\pi-\theta$ when the minor angle is θ.





Quote the formula first – the method mark follows the formula.

> **Worked example – a helipad from a major sector and a triangle** (adapted from Edexcel IAL WMA11 June 2025 Q6). A landing pad is the major sector AOB of a circle centre O joined to triangle AOC: $AO=OB=15$ m, $BC=2$ m, CBO is a straight line, and angle $ACO=0.6$ radians.

(a) Show angle $COA=1.847$ radians to 3 decimal places.

- Known: $OC=OB+BC=15+2=17$ m; in triangle AOC, side $AO=15$ is opposite the given angle 0.6, and side $OC=17$ is opposite angle CAO.

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- Sine rule (quote it first):

$$\frac{\sin CAO}{17} = \frac{\sin 0.6}{15} \implies \sin CAO = \frac{17\sin 0.6}{15} = 0.6399\ldots \implies CAO = 0.6944\ldots$$

$$COA = \pi - 0.6 - 0.6944\ldots = 1.8471\ldots = 1.847 \text{ (3 d.p.)}$$

- Check: angles sum to π: $0.6 + 0.6944 + 1.8471 \approx 3.1416$ ✓

(b) Total area of the pad.

- Major sector angle: $2\pi - 1.847 = 4.436$ radians.

$$\text{sector} = \tfrac{1}{2}r^2\theta = \tfrac{1}{2} \times 15^2 \times 4.436 = 499.1 \text{ m}^2 \text{ (awrt 499)}$$

$$\text{triangle} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2} \times 15 \times 17 \times \sin 1.847 = 122.7 \text{ m}^2$$

$$\text{total} = 499.1 + 122.7 = 622 \text{ m}^2 \text{ (3 s.f.)}$$

- Check: $\sin 1.847 = \sin(\pi - 1.847) = \sin 1.2946 = 0.962$ ✓

(c) Perimeter of the pad.

- The boundary is the major arc AB plus sides AC and BC.

$$\text{arc} = r\theta = 15 \times 4.436 = 66.5 \text{ m}$$

- Cosine rule for AC (quote it first):

$$AC^2 = 15^2 + 17^2 - 2(15)(17)\cos 1.847 = 225 + 289 + 139.1 = 653.1 \implies AC = 25.6 \text{ m}$$

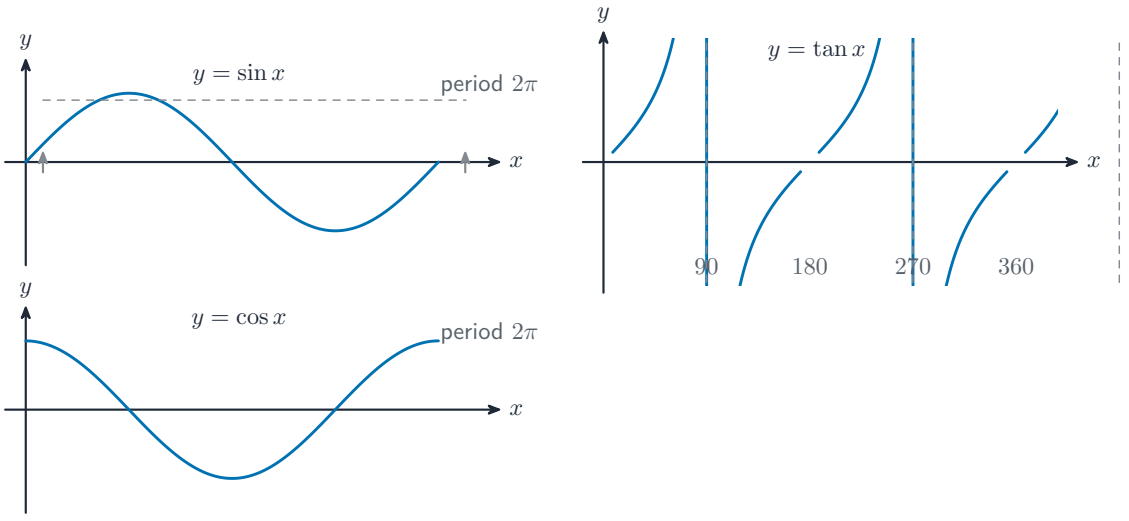
$$\text{perimeter} = 66.5 + 25.6 + 2 = 94.1 \text{ m (3 s.f.)}$$

- Check: $\cos 1.847 < 0$ (the angle is obtuse), so the $-2bc\cos A$ term adds ✓

■ Sine, cosine and tangent graphs

$y = \sin x$ and $y = \cos x$ repeat every 2π (360°); $y = \tan x$ repeats every π (180°) with asymptotes at $x = 90° + 180°n$. Know the symmetries: cos is even, $\sin(180° - x) = \sin x$, and $\tan x = \sin x / \cos x$. Curves such as $y = 3\sin x$, $y = \sin 2x$ and $y = \sin\left(x + \frac{\pi}{6}\right)$ follow from the transformation rules above. Solve equations **graphically** by counting intersections in a given interval.

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Count periods first, then solutions per period.

Worked example – counting solutions (adapted from Edexcel IAL WMA11 June 2025 Q4). For $y = 4\cos x$ with x in degrees: how many solutions does $5 + 4\cos x = 1$ have for $-720° < x \leq 720°$?

- Rearrange: $4\cos x = -4$, so $\cos x = -1$, giving $x = 180° + 360°n$.
- In $-720° < x \leq 720°$: $x = -540°, -180°, 180°, 540°$ – **4 solutions**.
- Check: $\cos(\pm 180°) = -1$ ✓, and $5 + 4(-1) = 1$ ✓. The interval is half-open, so $x = -720°$ (where $\cos = 1$) is excluded correctly.

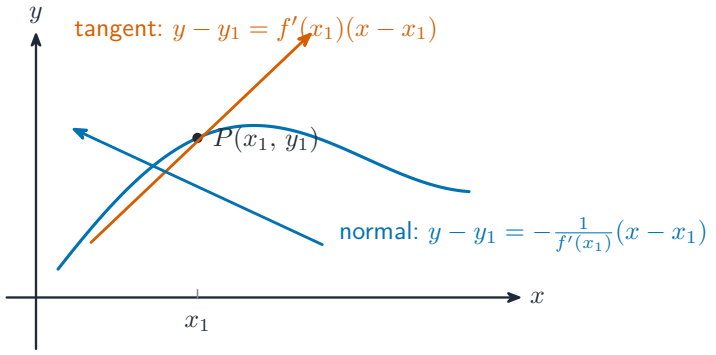
Differentiation

The **derivative** 导数 $f'(x) = \dfrac{dy}{dx}$ is the **gradient of the tangent** 切线斜率 to $y = f(x)$ at a point – the limit of the chord gradients as the two points merge. It is the **rate of change** 变化率 of y with respect to x. The **second derivative** 二阶导数 $f''(x)$ differentiates again. (The chain rule is **not** required in P1.)

Differentiate term by term with the power rule, expanding products and splitting fractions first:

$$y = (2x+5)(x-1) = 2x^2 + 3x - 5 \implies \frac{dy}{dx} = 4x + 3 \qquad y = \frac{3x^2+5}{x^2} = 3 + 5x^{-2} \implies \frac{dy}{dx} = -10x^{-3}$$

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Tangent gradient $= f'(x_1)$*; for nonzero finite tangent gradient, normal gradient* $= -1/f'(x_1)$*. A horizontal tangent has a vertical normal.*

Worked example – gradient condition on a curve (adapted from Edexcel IAL WMA11 June 2025 Q2). C: $y = 2x^{5/2} - 4x + 3$; the point P on C has x-coordinate 2^k and the gradient at P is 16. Find k.

- Differentiate (powers drop by one): $\dfrac{dy}{dx} = 2 \cdot \frac{5}{2}x^{3/2} - 4 = 5x^{3/2} - 4$.
- Set the gradient to 16 at $x = 2^k$:

$$5(2^k)^{3/2} - 4 = 16 \implies (2^k)^{3/2} = 4 = 2^2 \implies 2^{3k/2} = 2^2 \implies \tfrac{3k}{2} = 2 \implies k = \tfrac{4}{3}$$

- Check: $2^{4/3} = 2.5198\ldots$ and $5(2.5198)^{3/2} - 4 = 5(4.0) - 4 = 16$ ✓

Worked example – tangent, normal work and second derivative (adapted from Edexcel IAL WMA11 June 2025 Q8). A curve has $f'(x) = 3\sqrt{x} + kx^2$ and passes through $P(4, 12)$; the tangent at P is $y = 10x + c$.

(a) Show $k = \frac{1}{4}$ and find c.

- Known: the tangent's gradient equals $f'(4)$:

$$f'(4) = 3\sqrt{4} + 16k = 6 + 16k = 10 \implies k = \tfrac{1}{4}$$

- The tangent passes through $(4, 12)$: $12 = 10(4) + c \implies c = -28$.
- Check: $f'(4) = 6 + 16 \cdot \frac{1}{4} = 10$ ✓ and $10(4) - 28 = 12$ ✓

(b) $f''(x)$ at P.

$$f''(x) = \tfrac{3}{2}x^{-1/2} + 2kx = \tfrac{3}{2\sqrt{x}} + \tfrac{x}{2} \implies f''(4) = \tfrac{3}{4} + 2 = \tfrac{11}{4}$$

- Check: $\frac{3}{2\sqrt{4}} = \frac{3}{4}$ ✓

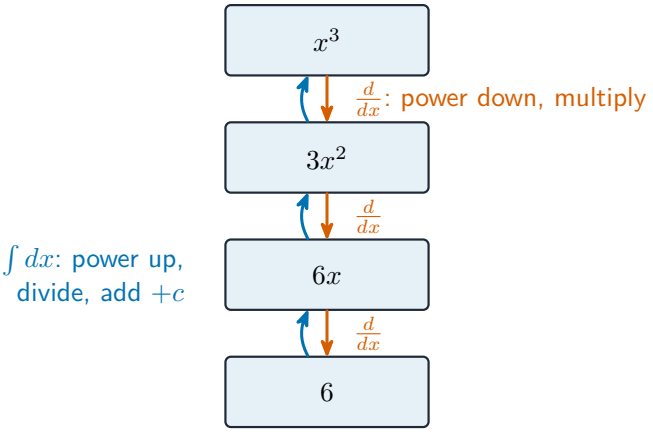
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Integration

Integration 积分 reverses differentiation: if $\dfrac{dy}{dx} = x^n$ $(n \neq -1)$, then

$$y = \int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

The **constant of integration** 积分常数 c is required – every vertical shift shares the same derivative. Raise each power by one, divide by the new power, and integrate term by term after expanding. Given $f'(x)$ **and one point** on the curve, find c and recover $y = f(x)$ exactly.



Each integral climbs one rung and picks up an arbitrary constant: $\int 6 \, dx = 6x + c$, not just $6x$.

Differentiation slides down; integration climbs back up, plus c.

Worked example – expand before integrating (adapted from Edexcel IAL WMA11 June 2025 Q3). Find $\displaystyle\int \frac{(x+3)^2}{3\sqrt{x}} \, dx$.

- Expand the numerator: $(x+3)^2 = x^2 + 6x + 9$; the denominator is $3x^{1/2}$.
- Divide term by term (powers of x subtract):

$$\frac{x^2}{3x^{1/2}} = \tfrac{1}{3}x^{3/2} \qquad \frac{6x}{3x^{1/2}} = 2x^{1/2} \qquad \frac{9}{3x^{1/2}} = 3x^{-1/2}$$

- Integrate each term (power up one, divide by the new power):

$$\int \left(\tfrac{1}{3}x^{3/2} + 2x^{1/2} + 3x^{-1/2}\right) dx = \tfrac{1}{3} \cdot \tfrac{2}{5}x^{5/2} + 2 \cdot \tfrac{2}{3}x^{3/2} + 3 \cdot 2x^{1/2} + c$$

$$= \frac{2}{15}x^{5/2} + \frac{4}{3}x^{3/2} + 6\sqrt{x} + c$$

- Check by differentiating: $\frac{2}{15} \cdot \frac{5}{2}x^{3/2} = \frac{1}{3}x^{3/2}$ ✓, $\frac{4}{3} \cdot \frac{3}{2} = 2$ ✓, $6 \cdot \frac{1}{2} = 3$ ✓ – the integrand returns exactly.

Worked example – recover $f(x)$ from $f'(x)$ and a point (adapted from Edexcel IAL WMA11 June 2025 Q8, continued). With $k = \frac{1}{4}$, so $f'(x) = 3x^{1/2} + \frac{1}{4}x^2$, and $P(4, 12)$ on the curve:

$$f(x) = \int \left(3x^{1/2} + \tfrac{1}{4}x^2\right) dx = 3 \cdot \tfrac{2}{3}x^{3/2} + \tfrac{1}{4} \cdot \tfrac{1}{3}x^3 + d = 2x^{3/2} + \frac{x^3}{12} + d$$

- Substitute P: $2(4)^{3/2} + \frac{64}{12} + d = 12$, and $4^{3/2} = 8$:

$$16 + \tfrac{16}{3} + d = 12 \implies d = -\tfrac{28}{3} \implies f(x) = 2x^{3/2} + \frac{x^3}{12} - \frac{28}{3}$$

- Check: $f(4) = 16 + \frac{16}{3} - \frac{28}{3} = 16 - 4 = 12$ ✓, and differentiating the answer gives $f'(x) = 3x^{1/2} + \frac{x^2}{4}$, so $f'(4) = 6 + 4 = 10$ ✓ (the tangent gradient from before).

Checklist before you call this unit done

- Manipulate $a^{m/n}$, negative powers and surds exactly, and rationalise denominators.
- Read $b^2 - 4ac$ three ways, complete the square, and solve hidden quadratics by substitution.
- Solve linear–quadratic simultaneous equations by substitution and interpret them as intersections.
- Solve and sketch linear and quadratic inequalities with correct dotted/solid boundaries.
- Sketch cubics, reciprocals (with asymptotes) and transformed curves, naming direction, type and scale factor of each single transformation.
- Write line equations in both forms; use $m_1 m_2 = -1$ for perpendiculars.
- Quote sine rule, cosine rule and $\frac{1}{2}ab\sin C$; handle the ambiguous case and major sectors in radians.
- Count trigonometric-graph solutions over an interval before listing them.
- Differentiate and integrate x^n term by term, after expanding; always add $+c$.
- Turn $f'(x)$ plus one point into $f(x)$, and check by substituting the point back.

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2 · Pure Mathematics 2

Pearson Edexcel · International A-Level · Pure Mathematics

Pure Mathematics 2: building on the P1 toolkit

Pure Mathematics 2 (P2) is the second compulsory unit of International AS Mathematics and International AS Pure Mathematics, and of both International A Level routes. This reference covers every P2 topic in the Edexcel IAL specification (Issue 3, April 2019): proof, the factor and remainder theorems, circle geometry, sequences and series, exponentials and logarithms, trigonometric identities and equations, stationary points, and integration including areas and the trapezium rule.

- The P2 paper (WMA12/01) lasts **1 hour 30 minutes** with **75 marks**. A knowledge of P1 and its formulae is assumed and **may be tested** on the P2 paper.
- The same calculator and yellow *Mathematical Formulae and Statistical Tables* rules apply as for P1. The formulae below do **not** appear in the booklet – memorise them.
- "Show all stages of your working" questions reject calculator-only answers; give inexact answers to 3 significant figures unless told otherwise.

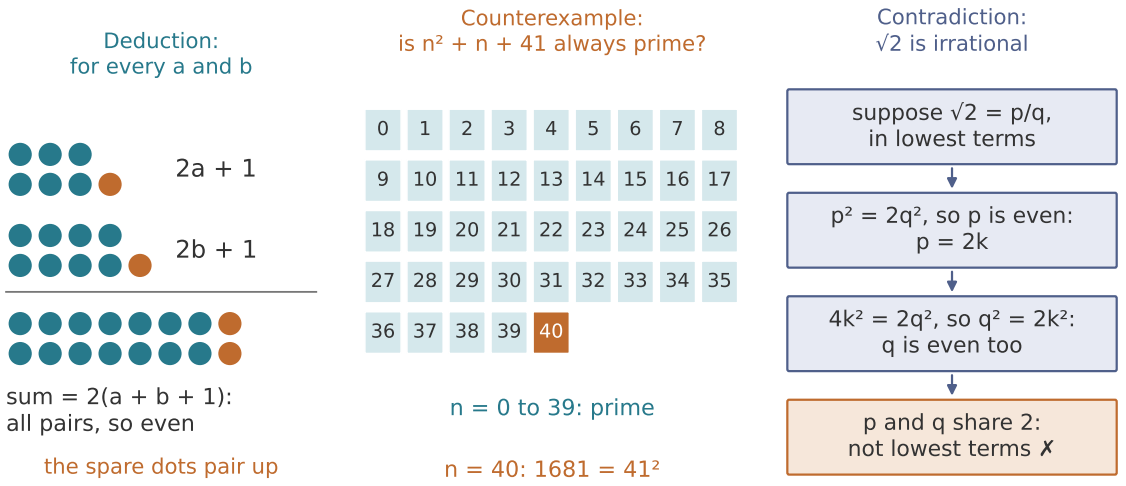
Formulae you must know for P2 (from the specification's assessment information):

Formula	Use
$\log_a x + \log_a y \equiv \log_a(xy)$, $\log_a x - \log_a y \equiv \log_a \dfrac{x}{y}$, $k \log_a x \equiv \log_a(x^k)$	laws of logarithms 对数运算法则
$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta \equiv 1$	trigonometric identities 三角恒等式
area under a curve $= \displaystyle\int_a^b y\,dx \quad (y \geq 0)$	definite integral as an area

Proof

A **proof** 证明 proceeds from given assumptions through logical steps to a conclusion. P2 uses three methods:

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Deduction proves every case at once; one counterexample ($n = 40$) disproves an "always" claim; contradiction assumes the opposite and breaks it

- **Proof by exhaustion** 穷举法: split the possibilities into cases and check every case.
- **Disproof by counterexample** 反例法: one example that breaks the claim disproves it.
- Algebraic proof: start from what is given, manipulate identities (not equations to solve), and end with the target statement.

Worked example – exhaustion (adapted from Edexcel IAL WMA12 June 2025 Q10). Prove that for all positive integers m that are not multiples of 3, the value of $m^2 + 3m + 2$ is always a multiple of 3.

- Factorise first: $m^2 + 3m + 2 = (m + 1)(m + 2)$.
- A positive integer not divisible by 3 is either $m = 3q + 1$ or $m = 3q + 2$ – those are all the cases, so exhausting them proves the claim.
- Case $m = 3q + 1$: $(m + 1)(m + 2) = (3q + 2)(3q + 3) = 3(3q + 2)(q + 1)$ – a multiple of 3.
- Case $m = 3q + 2$: $(m + 1)(m + 2) = (3q + 3)(3q + 4) = 3(q + 1)(3q + 4)$ – a multiple of 3.
- Both cases give a multiple of 3, so $3 \mid m^2 + 3m + 2$ whenever $3 \nmid m$. ■
- Check: $m = 1$: $6 = 3 \times 2$ ✓; $m = 2$: $12 = 3 \times 4$ ✓; $m = 4$: 30 ✓; $m = 5$: 42 ✓.

Worked example – counterexample. Disprove: "$n^2 - n + 1$ is prime for all integers $n \geq 2$."

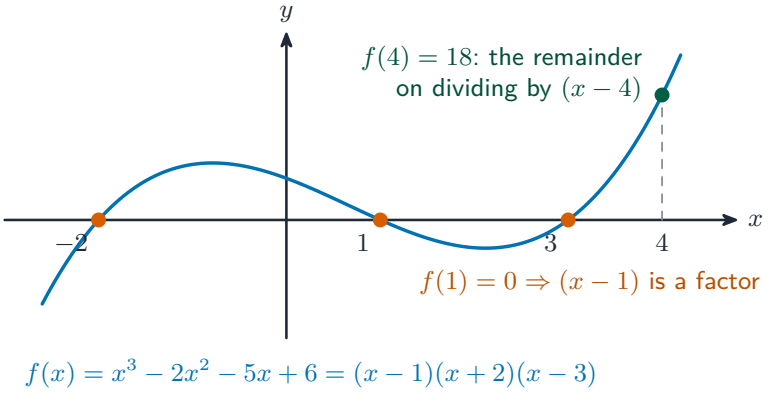
- One counterexample is enough. Try $n = 5$: $25 - 5 + 1 = 21 = 3 \times 7$, which is not prime.

- The claim is false. Check $n = 2, 3, 4$ give the primes $3, 7, 13$ – the pattern holds at first, which is exactly why a counterexample, not more examples, settles the question.

Algebra: the factor and remainder theorems

For a polynomial $f(x)$:

- **Factor theorem** 因式定理: $(x - b)$ is a factor of $f(x)$ exactly when $f(b) = 0$. For divisors of the form $(ax - b)$: if $f\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor.
- **Remainder theorem** 余项定理: dividing $f(x)$ by $(ax - b)$ leaves remainder $f\left(\frac{b}{a}\right)$.
- **Algebraic division** 代数除法 by $(ax \pm b)$ gives a **quotient** 商 and a **remainder**; the degree drops by one: $f(x) = (ax - b) \times (\text{quotient}) + \text{remainder}$.



Roots are factors; the height at $x = b$ is the remainder on dividing by $(x - b)$

> **Worked example – both theorems at once** (adapted from Edexcel IAL WMA12 June 2025 Q4). $f(x) = ax^3 + bx^2 + 5x - 3$, where $(x + 3)$ is a factor of $f(x)$ and dividing $f(x)$ by $(2x - 1)$ leaves remainder 7.

(a) Show that $b = 3a + 2$.

- Factor theorem: $f(-3) = 0$:

$$a(-27) + b(9) + 5(-3) - 3 = 0 \implies -27a + 9b - 18 = 0 \implies b = 3a + 2$$

(b) Find a and b.

- Remainder theorem with $(2x - 1)$: $f\left(\frac{1}{2}\right) = 7$:

$$\frac{a}{8} + \frac{b}{4} + \frac{5}{2} - 3 = 7 \implies \frac{a}{8} + \frac{b}{4} = 7.5 \implies a + 2b = 60$$

- Substitute $b = 3a + 2$: $a + 6a + 4 = 60$, so $a = 8$ and $b = 26$.

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- Check: $f(x) = 8x^3 + 26x^2 + 5x - 3$; $f(-3) = -216 + 234 - 15 - 3 = 0$ ✓ and $f(\frac{1}{2}) = 1 + 6.5 + 2.5 - 3 = 7$ ✓.

(c) Quotient and remainder when $f(x)$ is divided by $(x - 2)$.

- Divide $8x^3 + 26x^2 + 5x - 3$ by $(x - 2)$: quotient $8x^2 + 42x + 89$, remainder 175.
- Check by rebuilding: $(x - 2)(8x^2 + 42x + 89) + 175 = 8x^3 + 42x^2 + 89x - 16x^2 - 84x - 178 + 175 = 8x^3 + 26x^2 + 5x - 3$ ✓. The remainder also equals $f(2) = 64 + 104 + 10 - 3 = 175$ ✓.

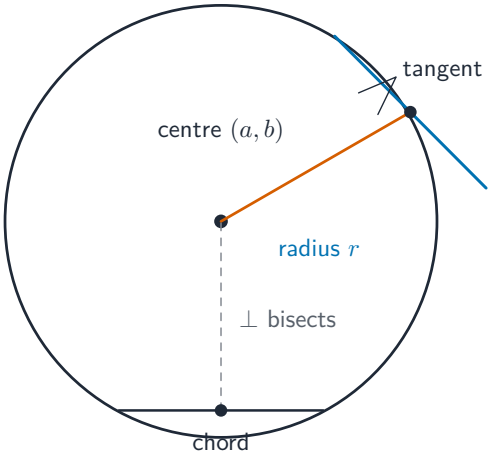
Coordinate geometry of the circle

A circle centre (a, b) radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Given the expanded form $x^2 + y^2 + fx + gy + c = 0$, complete the square to recover centre $\left(-\frac{f}{2}, -\frac{g}{2}\right)$ and $r^2 = \frac{f^2+g^2}{4} - c$. Three circle properties are examined:

- the angle in a semicircle is a right angle;
- the perpendicular from the centre to a **chord** 弦 bisects the chord;
- a **tangent** 切线 is perpendicular to the radius at the point of contact.



Tangent ⊥ radius at the contact point; the perpendicular from the centre to a chord **bisects** it; the angle in a semicircle is 90.

Radius ⊥ tangent; centre-to-chord perpendicular bisects the chord.

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Worked example – diameter endpoints (adapted from Edexcel IAL WMA12 June 2025 Q2). The line joining $(-2, 5)$ and $(4, 15)$ is a diameter of circle C.

(a) Equation of C.

- The centre is the midpoint: $\left(\frac{-2+4}{2}, \frac{5+15}{2}\right) = (1, 10)$.
- The diameter squared: $d^2 = (4-(-2))^2 + (15-5)^2 = 36 + 100 = 136$, so $r^2 = 136 \div 4 = 34$.

$$(x-1)^2 + (y-10)^2 = 34$$

- Check: both endpoints satisfy the equation: $(-3)^2+(-5)^2 = 34$ ✓ and $3^2+5^2 = 34$ ✓.

(b) Exact coordinates of the point on C nearest the x-axis.

- The nearest point lies on the vertical radius: same x as the centre, r below it:

$$(1,\ 10-\sqrt{34})$$

- Check: $(1-1)^2 + (10-\sqrt{34}-10)^2 = 34$ ✓, and no smaller y is possible since the centre sits $10 > \sqrt{34}$ above the axis.

Sequences and series

A **sequence** 数列 may be given by a formula for the nth term or by a relation $x_{n+1} = f(x_n)$. Describe it as **increasing** 递增, **decreasing** 递减 or **periodic** 周期 (from its terms, not its formula).



Romanesco broccoli: each bud repeats the whole at a smaller scale, a geometric pattern

Arithmetic sequences 等差数列: constant difference d.

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$$a_n = a + (n-1)d \qquad S_n = \frac{n}{2}\big(2a + (n-1)d\big) \qquad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

The proof of the sum formula (write S_n forwards and backwards, add) should be known, as should Σ notation.

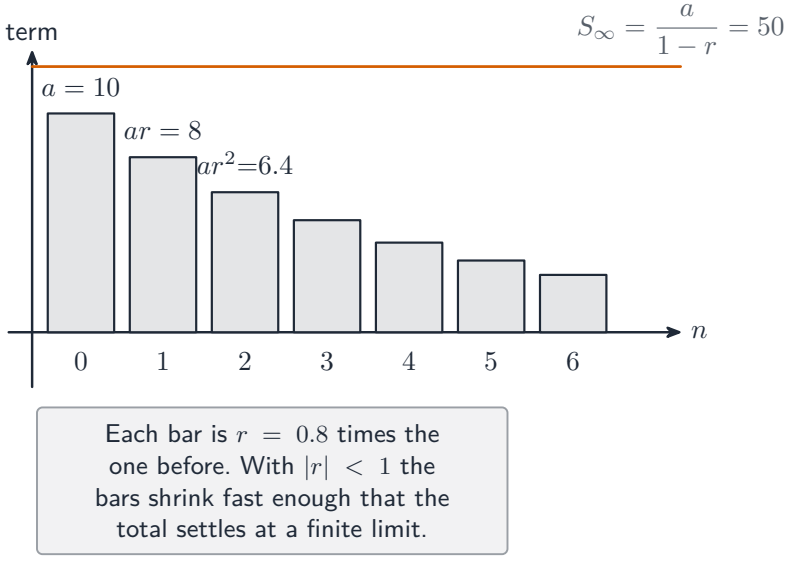
Geometric sequences 等比数列: constant ratio r.

$$a_n = ar^{n-1} \qquad S_n = \frac{a(1-r^n)}{1-r} \qquad S_\infty = \frac{a}{1-r}\ (|r|<1)$$

The sum formula's proof (telescoping $S_n - rS_n$) should be known. Use **logarithms** to solve for n when a term value is given.

Binomial expansion 二项展开式 for a positive integer n:

$$(a+bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \cdots + (bx)^n, \qquad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$



$|r| < 1$ means the terms shrink; the sum settles at S_∞.

Worked example – geometric series with logs (adapted from Edexcel IAL WMA12 June 2025 Q7). A geometric series begins $10 + 8 + 6.4 + \cdots$

(a) Sum to infinity.

- $a = 10$, $r = 8/10 = 0.8$, and $|r| < 1$:

$$S_\infty = \frac{a}{1-r} = \frac{10}{1-0.8} = 50$$

(b) The kth term is less than 0.0005 – find the smallest k, by algebra.

- The kth term is $10(0.8)^{k-1}$:

$$10(0.8)^{k-1} < 0.0005 \implies (0.8)^{k-1} < 0.00005$$

- Take logs (the log of a number below 1 is negative – keep the inequality direction when dividing by it):

$$(k-1)\log 0.8 < \log 0.00005 \implies k - 1 > \frac{\log 0.00005}{\log 0.8} = 44.38\ldots$$

- So $k > 45.38$ and the smallest integer is $k = 46$.
- Check: $10(0.8)^{45} = 0.000436\ldots < 0.0005$ ✓ but $10(0.8)^{44} = 0.000544\ldots > 0.0005$ ✓.

Worked example – maximum arithmetic sum (adapted from the same question). An arithmetic series begins $850 + 843 + 836 + \cdots$ ($d = -7$). Its greatest partial sum:

- Terms stay positive until $u_n = 850 - 7(n-1) > 0$, so $n \le 122.4$; the 122nd term is $850 - 7(121) = 3 > 0$ and the 123rd is $-4 < 0$.

$$S_{122} = \frac{122}{2}\big(2(850) + 121(-7)\big) = 61(1700 - 847) = 61 \times 853 = 52\,033$$

- Check: adding a negative 123rd term would reduce the sum, so S_{122} is the maximum ✓.

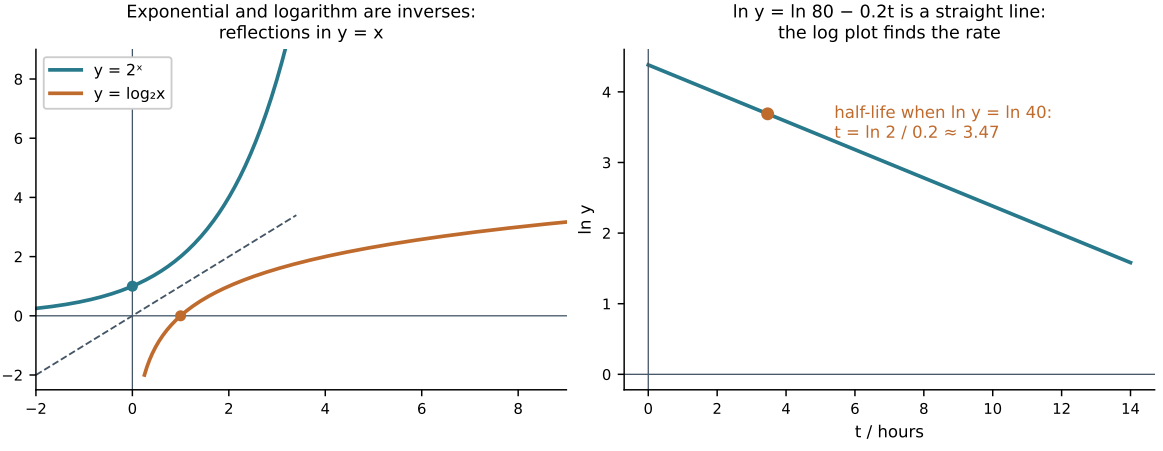
Exponentials and logarithms

$y = a^x$ ($a > 0$, $a \neq 1$) passes through $(0, 1)$ and grows without bound for $a > 1$. The **logarithm** 对数 $\log_a x$ inverts it: $\log_a x = y$ means $a^y = x$. Know:

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A seismograph: earthquake magnitude is a logarithmic scale, so each step up is ten times the shaking



Exponential and log are inverse functions; ln y against t is a straight line, and the half-life is ln 2 / 0.2 ≈ 3.47 hours

$$\log_a x + \log_a y \equiv \log_a(xy) \qquad \log_a x - \log_a y \equiv \log_a \frac{x}{y} \qquad k\log_a x \equiv \log_a(x^k) \qquad \log_a a = 1$$

and $\log_a \frac{1}{x} = -\log_a x$. Solve $a^x = b$ with $x = \log_a b$ (a change of base may be used). Always check candidates against the **domain** 定义域: logs need positive arguments.

> **Worked example – a log equation** (adapted from Edexcel IAL WMA12 June 2025 Q6). Solve $2\log_2(4-x) = 3 + \log_2\left(\dfrac{x+11}{2}\right)$.

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- Bring everything to one log per side:

$$\log_2(4-x)^2 = \log_2 8 + \log_2\left(\frac{x+11}{2}\right) = \log_2\left(4(x+11)\right)$$

- Equal logs give equal arguments:

$$(4-x)^2 = 4(x+11) \implies x^2 - 8x + 16 = 4x + 44 \implies x^2 - 12x - 28 = 0$$

$$(x-14)(x+2) = 0 \implies x = 14 \text{ or } x = -2$$

- Domain check: $4 - x > 0$ needs $x < 4$, so $x = 14$ is rejected. $x = -2$.
- Check: LHS $= 2\log_2 6$; RHS $= 3 + \log_2(4.5) = \log_2(8 \times 4.5) = \log_2 36 = 2\log_2 6$ ✓.

Worked example – exact intersection in $\log_3$ form (same question). The curves $y = 3^{2x+1}$ and $y = 6 \times 3^x$ meet at P; write P as $(\log_3 a, b)$.

- Set equal and substitute $u = 3^x$:

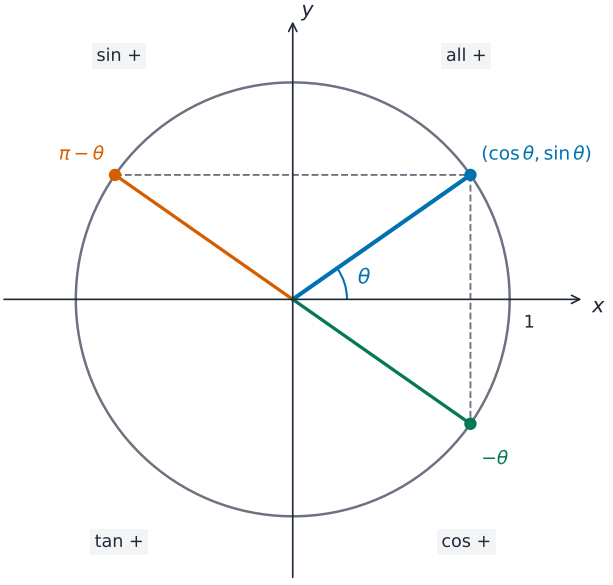
$$3 \cdot 3^{2x} = 6 \cdot 3^x \implies 3u^2 = 6u \implies u = 2 \implies 3^x = 2 \implies x = \log_3 2$$

$$y = 6 \times 3^x = 6 \times 2 = 12$$

- $P = (\log_3 2,\ 12)$, so $a = 2$, $b = 12$. Check: $3^{2\log_3 2+1} = 3 \cdot 3^{2\log_3 2} = 3 \cdot 4 = 12$ ✓.

Trigonometric identities and equations

Two identities convert between sine, cosine and tangent:



Reflections in the unit circle give the symmetry rules

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$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta \equiv 1$$

Solve **trigonometric equations** 三角方程 in a given interval by reducing to one function (often a hidden quadratic), then reading the second solution from the graph's symmetry. State the interval, work in its units (degrees or radians), and reject values outside.

Worked example – radians and exact fractions (adapted from Edexcel IAL WMA12 June 2025 Q8). Solve $3\tan\left(2x + \frac{\pi}{5}\right) = \sqrt{3}$ for $0 \le x < \pi$.

- Divide by 3: $\tan\left(2x + \frac{\pi}{5}\right) = \frac{1}{\sqrt{3}}$, and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$:

$$2x + \frac{\pi}{5} = \frac{\pi}{6} + k\pi \implies 2x = -\frac{\pi}{30} + k\pi \implies x = -\frac{\pi}{60} + \frac{k\pi}{2}$$

- In $0 \le x < \pi$: $x = \dfrac{29\pi}{60}$ (k = 1) and $x = \dfrac{59\pi}{60}$ (k = 2).
- Check: at $x = \frac{29\pi}{60}$: $2x + \frac{\pi}{5} = \frac{7\pi}{6}$ and $\tan\frac{7\pi}{6} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ ✓.

Worked example – hidden quadratic in cosine (same question). Solve $5\sin\theta\tan\theta = \cos\theta + 4$ for $0 \le \theta < 360°$.

- Replace $\tan\theta = \sin\theta/\cos\theta$ and multiply through by $\cos\theta$:

$$\frac{5\sin^2\theta}{\cos\theta} = \cos\theta + 4 \implies 5\sin^2\theta = \cos^2\theta + 4\cos\theta$$

- Use $\sin^2\theta = 1 - \cos^2\theta$:

$$5(1 - \cos^2\theta) = \cos^2\theta + 4\cos\theta \implies 6\cos^2\theta + 4\cos\theta - 5 = 0$$

- Quadratic formula in $\cos\theta$:

$$\cos\theta = \frac{-4 \pm \sqrt{16 + 120}}{12} = \frac{-2 \pm \sqrt{34}}{6}$$

- The $-$ root is below -1 (reject); $\cos\theta = 0.6385$ gives $\theta = 50.3°$ and, from cosine's symmetry, $\theta = 360° - 50.3° = 309.7°$.
- Check: $\cos 50.3° = 0.638$ ✓; both angles lie in the interval ✓.

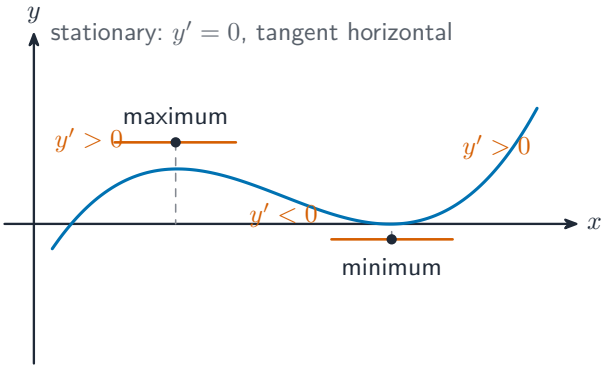
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Differentiation: stationary points

Stationary points 驻点 have $\dfrac{dy}{dx} = 0$. Classify with the sign of the derivative:

- derivative changes $+$ to $-$: **maximum** 极大值;
- derivative changes $-$ to $+$: **minimum** 极小值;
- the second derivative helps: $f''(x) > 0$ at the point confirms a minimum, $f''(x) < 0$ a maximum.

A function is **increasing** where $f'(x) > 0$ and **decreasing** where $f'(x) < 0$. Practical problems – least material, least cost – translate into minimising an expression in one variable.



Stationary means the tangent is horizontal; the sign change classifies it.

Worked example – a modelling minimum (adapted from Edexcel IAL WMA12 June 2025 Q9). An open trough has right-triangle ends with legs $3x$ and $2x$ (metres), rectangular sides $3x \times L$ and $2x \times L$, and capacity 12 m^3. The metal area is S m^2.

(a) Show $S = 6x^2 + \dfrac{20}{x}$.

- Volume: cross-section $\times$ length $= \frac{1}{2}(3x)(2x) \times L = 3x^2L = 12$, so $L = \dfrac{4}{x^2}$.
- Surface: two triangles plus the two rectangles:

$$S = 2 \times \tfrac{1}{2}(3x)(2x) + 3xL + 2xL = 6x^2 + 5x \cdot \frac{4}{x^2} = 6x^2 + \frac{20}{x}$$

(b) Minimum S by calculus.

$$\frac{dS}{dx} = 12x - \frac{20}{x^2} = 0 \implies 12x^3 = 20 \implies x^3 = \frac{5}{3} \implies x = 1.1856\ldots$$

$$S = 6(1.1856)^2 + \frac{20}{1.1856} = 8.43 + 16.87 = 25.3 \text{ m}^2 \text{ (1 d.p.)}$$

(c) Justify the minimum.

$$\frac{d^2S}{dx^2} = 12 + \frac{40}{x^3} > 0 \quad \text{for all } x > 0$$

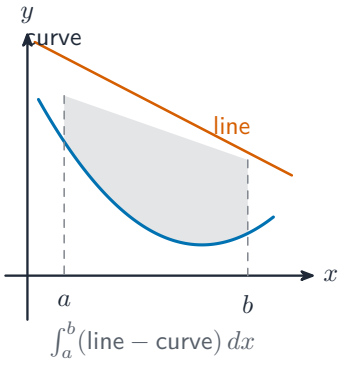
- Positive second derivative everywhere on the domain: the stationary point is a minimum ✓.
- Check: at $x = 1.1856$, $\frac{d^2S}{dx^2} = 12 + 40/(5/3) = 12 + 24 = 36 > 0$ ✓.

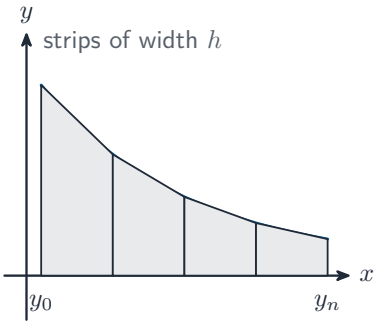
Integration: areas and the trapezium rule

Evaluate **definite integrals** 定积分 with the fundamental theorem:

$$\int_a^b y\,dx = F(b) - F(a) \quad \text{where } F'(x) = y$$

The definite integral is the **area under the curve** 曲线下面积 for $y \geq 0$. For a region bounded by a curve and a line, integrate (upper − lower) between their intersections, splitting wherever their order changes. Check which graph is above the other.





Area between uses $\int$(upper − lower), splitting at crossings; the trapezium rule joins neighbouring values with straight strips.

Worked example – area between curve and line (adapted from Edexcel IAL WMA12 June 2025 Q3). C: $y = 3x + \dfrac{16}{x^2} - 8$ $(x > 0)$ with $P(1, 11)$ and $Q(4, 5)$ on C; find the area between C and the line segment PQ.

- Line PQ: gradient $\dfrac{5-11}{4-1} = -2$, so $y - 11 = -2(x-1) \implies y = -2x + 13$.
- Check Q: $-8 + 13 = 5$ ✓.
- Area under the line: $\displaystyle\int_1^4 (-2x + 13)\,dx = \left[-x^2 + 13x\right]_1^4 = 36 - 12 = 24$.

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- Area under the curve: $\int_1^4 \left(3x + 16x^{-2} - 8\right) dx = \left[\frac{3}{2}x^2 - \frac{16}{x} - 8x\right]_1^4 = (24 - 4 - 32) - (\frac{3}{2} - 16 - 8) = 10.5$.
- Region area: $24 - 10.5 = \mathbf{13.5}$ square units.
- Check: the line is above the curve throughout (at $x = 2$: line 9, curve $3(2)+4-8 = 2$), so subtracting curve from line is the right way round ✓.

The trapezium rule 梯形法则 approximates an area with n strips of width h:

$$\int_a^b y\,dx \approx h\left[\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}\right] \quad \text{or} \quad \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})\right]$$

More strips (or a straighter curve) improve the estimate; a curve that bends upwards (**convex**, U-shaped like x^2) leaves every chord **above** it, so the rule **overestimates**, while a curve that bends downwards (**concave**, like $\sqrt{x}$ or a circle arc) leaves chords **below** it, so the rule **underestimates**.

Worked example – trapezium rule with a table. Estimate $\int_1^3 \frac{1}{x}\,dx$ with 4 strips.

- Four strips over $[1, 3]$ give $h = 0.5$ and five values of $y = \dfrac{1}{x}$: at $x = 1,\ 1.5,\ 2,\ 2.5,\ 3$ the values are $1,\ 0.6667,\ 0.5,\ 0.4,\ 0.3333$.

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.5}{2}\left[1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)\right] = 0.25 \times 4.4667 = 1.12$$

- The true value is $\ln 3 = 1.099$ (3 s.f.). The estimate is an **overestimate**: $y = 1/x$ is convex (it bends upwards), so every straight strip sits above the curve ✓.

Checklist before you call this unit done

- Structure a proof; use exhaustion over complete cases; disprove with one counterexample.
- Apply the factor and remainder theorems, including $(ax - b)$ divisors, and carry out algebraic division with quotient and remainder.
- Find a circle's centre and radius from either form; use the three circle properties.
- Use a_n, S_n and S_∞ for arithmetic and geometric series (both sum-formula proofs), Σ notation, and logs to find n.
- Expand $(a + bx)^n$ with binomial coefficients, and match given coefficients.
- Solve log equations with the three laws, checking domains; give intersections exactly as $(\log_a b, c)$ when asked.
- Reduce trigonometric equations with $\tan = \sin / \cos$ and $\sin^2 + \cos^2 = 1$; solve in degrees or radians as given, listing all interval solutions.

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- Find and classify stationary points; justify with a sign change or second derivative; set up modelling minima in one variable.
- Evaluate definite integrals; find areas under curves and between curve and line; apply the trapezium rule from a table and state the direction of its error.

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3 · Pure Mathematics 3

Pearson Edexcel · International A-Level · Pure Mathematics

Pure Mathematics 3: functions, identities and calculus at speed

Pure Mathematics 3 (P3) extends P1 and P2: rational functions and inverses, the modulus, combined transformations, reciprocal trigonometric functions and harmonic form, log-graph model fitting, the full differentiation rules, integration by recognition, and numerical methods. P1 and P2 knowledge is assumed and may be tested on the WMA13/01 paper (1 hour 30 minutes, 75 marks; same calculator and formula-booklet rules).

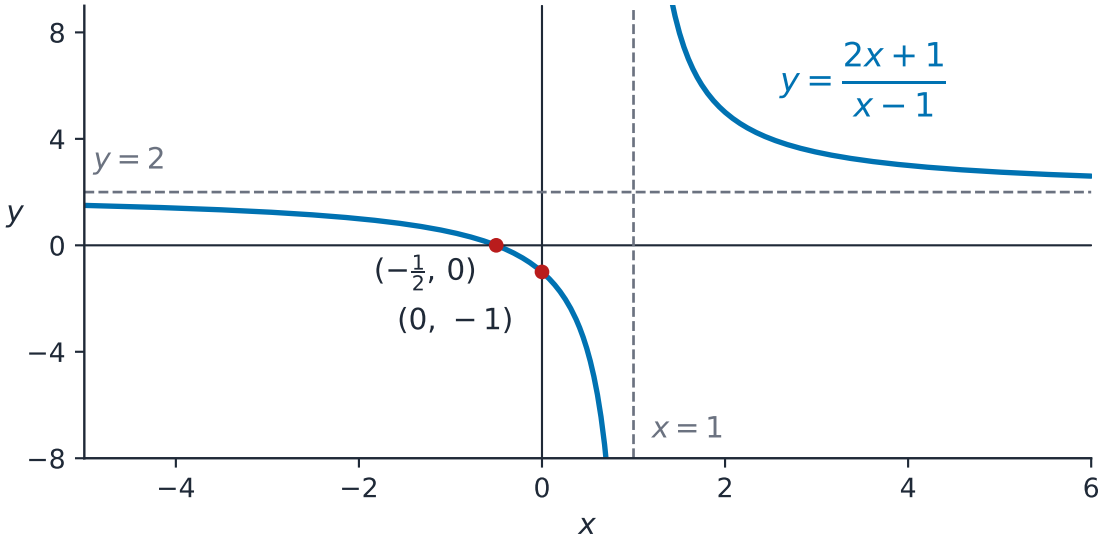
Formulae you must know for P3 (not in the booklet):

Family	Formulae
compound angle	$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$; $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
double angle	$\sin 2A \equiv 2 \sin A \cos A$; $\cos 2A \equiv \cos^2 A - \sin^2 A$; $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$
reciprocal identities	$\sec^2 A \equiv 1 + \tan^2 A$; $\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$
derivatives	$\sin kx \to k \cos kx$; $\cos kx \to -k \sin kx$; $\tan kx \to k \sec^2 kx$; $e^{kx} \to k e^{kx}$; $\ln x \to \frac{1}{x}$; $a^x \to a^x \ln a$
integrals	$\int \cos kx = \frac{1}{k} \sin kx$; $\int \sin kx = -\frac{1}{k} \cos kx$; $\int e^{kx} = \frac{1}{k} e^{kx}$; $\int \frac{1}{x} = \ln\lvert x \rvert$

Algebra: rational expressions and functions

Simplify rational expressions 有理式 by factorising and cancelling; **algebraic division** serves denominators up to quadratics. Only cancel factors, never terms.

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Asymptotes first, then intercepts, then the two branches

A **function** 函数 is a one-one or many-one mapping; its **domain** 定义域 is the allowed inputs and its **range** 值域 the outputs. The **composition** 复合 fg means "do g first, then f". The **inverse** 反函数 f^{-1} undoes f (it exists where f is one-one), and $f^{-1}f(x) = ff^{-1}(x) = x$; the graph of f^{-1} reflects the graph of f in $y = x$.

Worked example – inverse and composition (adapted from Edexcel IAL WMA13 June 2025 Q1). $f(x) = \dfrac{2x}{3x+1}$, $x \geq 0$.

(a) Range of f.

- At $x = 0$: $f(0) = 0$; as $x \to \infty$: $f(x) \to \frac{2}{3}$ from below, and f is increasing (larger x gives larger $f(x)$).

$$0 \leq f(x) < \tfrac{2}{3}$$

(b) Find $f^{-1}(x)$.

- Let $y = \dfrac{2x}{3x+1}$ and solve for x:

$$y(3x+1) = 2x \implies 3xy + y = 2x \implies x(3y-2) = -y \implies x = \frac{y}{2-3y}$$

$$f^{-1}(x) = \frac{x}{2-3x}$$

- Check: $f\left(\frac{x}{2-3x}\right) = \dfrac{2x/(2-3x)}{3x/(2-3x)+1} = \dfrac{2x}{3x+2-3x} = x$ ✓.

(c) Solve $f^{-1}(x) = f(x)$.

$$\frac{x}{2-3x} = \frac{2x}{3x+1} \implies x(3x+1) = 2x(2-3x) \implies x\big(3x+1-4+6x\big) = 0$$

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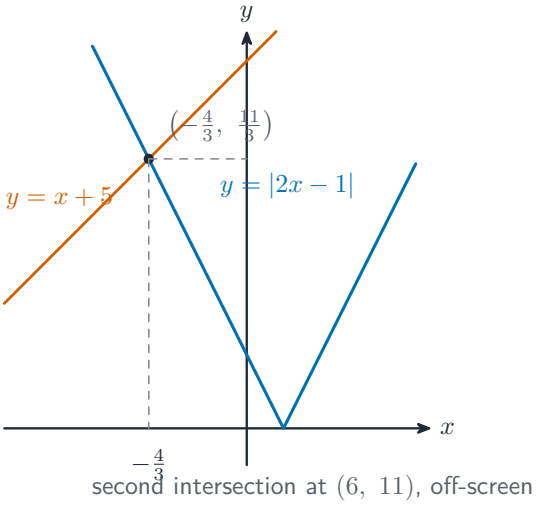
$$x(9x-3)=0 \implies x=0 \text{ or } x=\tfrac{1}{3}$$

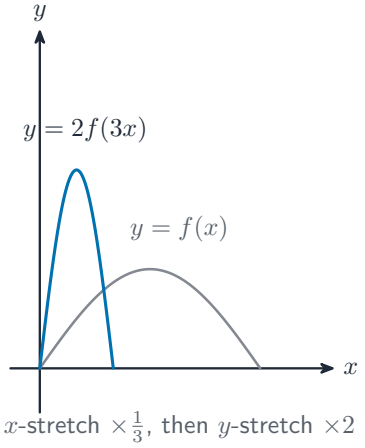
- Check: $f(\tfrac{1}{3})=\frac{2/3}{2}=\tfrac{1}{3}=f^{-1}(\tfrac{1}{3})$ ✓.

The modulus function and combined transformations

The **modulus** 绝对值 $|x|$ is x for $x\geq 0$ and $-x$ for $x<0$: $y=|ax+b|$ is the line $y=ax+b$ with its negative part reflected above the axis. Given $y=f(x)$, sketch $y=|f(x)|$ (reflect any below-axis part) and $y=f(|x|)$ (keep the right half, reflect it left). Solve modulus equations graphically or by cases, testing each solution.

Combined transformations 组合变换 apply the P1 rules together, for example $y=2f(3x)$ or $y=f(-x)+1$: stretches multiply, translations then shift; the order given in the syllabus is $y=af(x)$, $y=f(x)+a$, $y=f(x+a)$, $y=f(ax)$ combined in one expression ($y=f(ax+b)$ as such is not required).





The V meets the line at the marked point; the second intersection (6, 11) lies off-screen. Combine transformations one factor at a time.

Worked example – modulus equation (specification's own example). Solve $|2x-1|=x+5$.

- Case $2x-1\geq 0$ ($x\geq\frac{1}{2}$): $2x-1=x+5 \implies x=6$ ✓ (satisfies $x\geq\frac{1}{2}$).
- Case $2x-1<0$: $-(2x-1)=x+5 \implies -3x=4 \implies x=-\frac{4}{3}$ ✓ ($<\frac{1}{2}$).
- Both solutions are $x=6$ and $x=-\frac{4}{3}$. Check: $|2(6)-1|=11=6+5$ ✓; $|-\frac{8}{3}-1|=\frac{11}{3}=-\frac{4}{3}+5$ ✓.

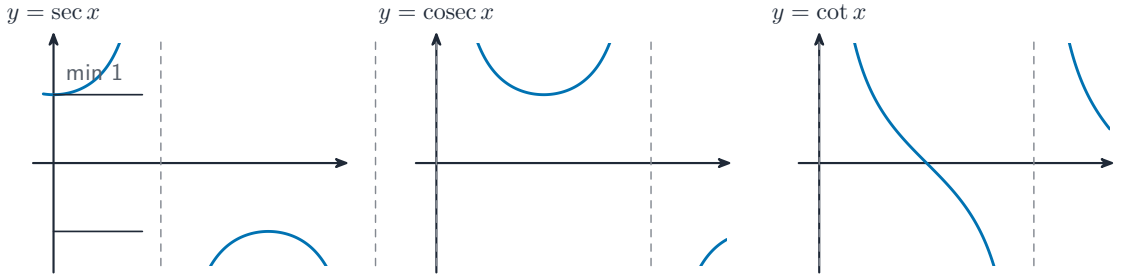
Further trigonometry: sec, cosec, cot and harmonic form

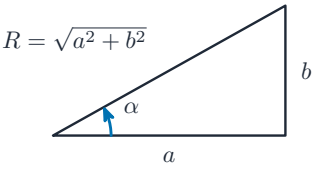
The **reciprocal functions** 倒数三角函数:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Cotangent is defined where $\sin\theta \neq 0$ and equals $1/\tan\theta$ where both expressions are defined. Use the identities $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$. Their graphs have **asymptotes** 渐近线 where the denominator function is zero. The **inverse functions** arcsin, arccos, arctan need **restricted domains** 限制定义域: $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos : [-1,1] \to [0,\pi]$, $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

Harmonic form 辅助角: $a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ with $R = \sqrt{a^2+b^2}$ with $R\cos\alpha = a$ and $R\sin\alpha = b$ – use both signs to choose the quadrant. If $R = 0$, the expression is zero. This form is ready to maximise or solve.





$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ with $\tan\alpha = \frac{b}{a}$: one amplitude R and one phase shift α replace both terms.

Harmonic form packs a cos θ + b sin θ into one cosine of a shifted angle.

Worked example – harmonic form (adapted from Edexcel IAL WMA13 June 2025 Q2). Express $f(x) = 7\cos x - 24\sin x$ in the form $R\cos(x+\alpha)$, $R > 0$, $0 < \alpha < 2\pi$.

- $R = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$.
- Expand $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$ and match: $25\cos\alpha = 7$ and $25\sin\alpha = 24$, so $\tan\alpha = \frac{24}{7}$ and

$$\alpha = 1.287 \text{ (3 d.p.)} \qquad f(x) = 25\cos(x + 1.287)$$

- Check: $25\cos(0+1.287) = 25(0.28) = 7.0$ ✓ and $-25\sin(1.287) = -24.0$ ✓.

Worked example – identity proof (adapted from Edexcel IAL WMA13 June 2025 Q8a). Prove $\tan 3x \equiv \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

- Write $3x = 2x + x$ and use the **compound angle** 复合角 formula:

$$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}, \qquad \tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2t}{1 - t^2} \ (t = \tan x)$$

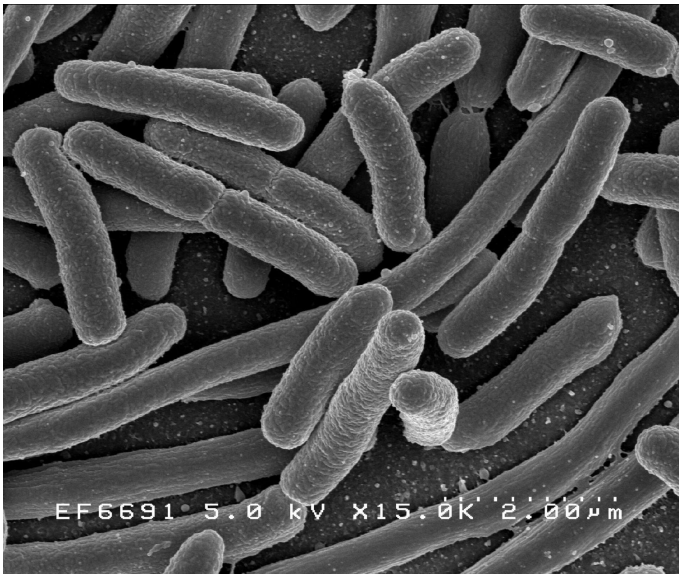
- Substitute:

$$\tan 3x = \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t^2}{1-t^2}} = \frac{2t + t(1-t^2)}{(1-t^2) - 2t^2} = \frac{3t - t^3}{1 - 3t^2} \qquad \blacksquare$$

- Check at $x = \frac{\pi}{6}$ ($t = \frac{1}{\sqrt{3}}$): numerator $\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}$; denominator $1 - 1 = 0$ – and $\tan\frac{\pi}{2}$ is undefined ✓ consistent.

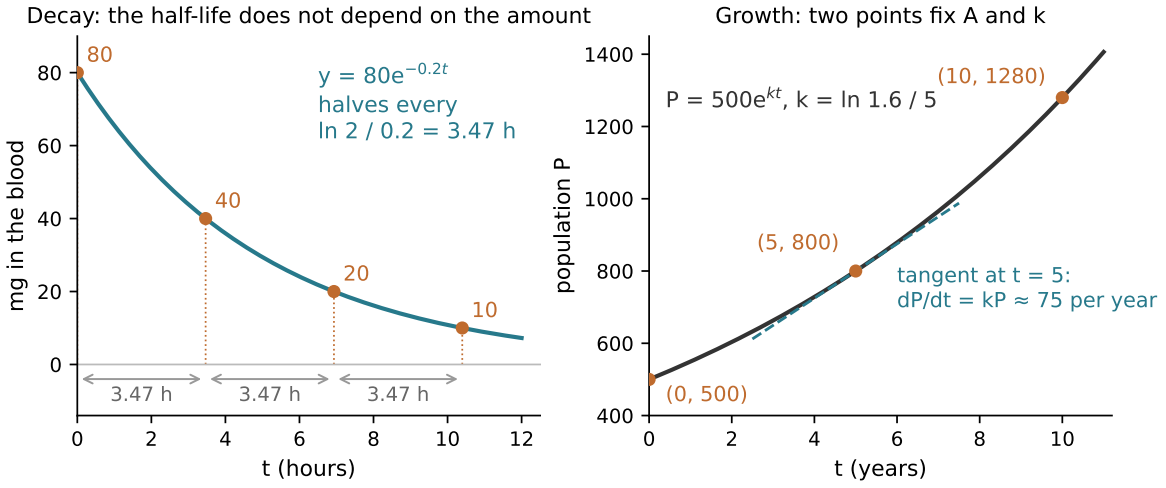
Exponentials, logarithms and log-graph models

$y = e^x$ passes through $(0, 1)$ and grows without bound; $\ln x$ is its inverse with domain $x > 0$ and the y-axis as asymptote. Solve $e^{ax+b} = p$ by taking ln, and $\ln(ax + b) = q$ by exponentiating.



Bacterial growth is roughly exponential while resources last; a log graph turns it into a straight line

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Decay halves in equal times whatever the amount; growth multiplies by 1.6 every 5 years, and the gradient kP is the rate of change

Log-graph models 对数图模型: for $y = ax^n$, taking logs gives $\log y = n \log x + \log a$ – plot $\log y$ against $\log x$ for a straight line with gradient n and intercept $\log a$. For $y = kb^x$, plot $\log y$ against x: gradient $\log b$, intercept $\log k$. Real logarithms require positive arguments and a valid common base; the displayed power-law form uses $a > 0, x > 0$, and the exponential form uses $k > 0, b > 0$.

> **Worked example – recover a power law** (adapted from Edexcel IAL WMA13 June 2025 Q3). A $\log_{10} y$–$\log_{10} x$ line passes through $(-3.5, 0)$ and $(0, -2)$. Express y as px^q.

- Gradient: $\dfrac{0-(-2)}{-3.5-0} = -\dfrac{4}{7}$; intercept -2:

$$\log_{10} y = -\tfrac{4}{7} \log_{10} x - 2$$

- Convert: $y = 10^{-\frac{4}{7}\log_{10} x - 2} = 10^{-2}\, x^{-4/7}$, so

$$y = \frac{1}{100}\, x^{-4/7} \qquad (p = \tfrac{1}{100},\ q = -\tfrac{4}{7})$$

- Check with a point on the line: $\log_{10} x = 0 \Rightarrow x = 1$, $\log_{10} y = -2 \Rightarrow y = \frac{1}{100}$ ✓ and the formula gives $y = \frac{1}{100} \cdot 1 = \frac{1}{100}$ ✓.

> **Worked example – exponential decay model** (adapted from Edexcel IAL WMA13 June 2025 Q6). A processor cools as $\theta = 21 + Ae^{-kt}$ (θ in °C, t in minutes), starting at 75 °C, reaching 25 °C after 5 minutes; find A, k, and the time when the temperature falls at 9 °C per minute.

- At $t = 0$: $75 = 21 + A$, so $A = 54$.
- At $t = 5$: $25 = 21 + 54e^{-5k} \implies e^{-5k} = \frac{4}{54} = \frac{2}{27}$:

$$-5k = \ln \tfrac{2}{27} \implies k = \tfrac{1}{5} \ln \tfrac{27}{2} = 0.521 \text{ (3 s.f.)}$$

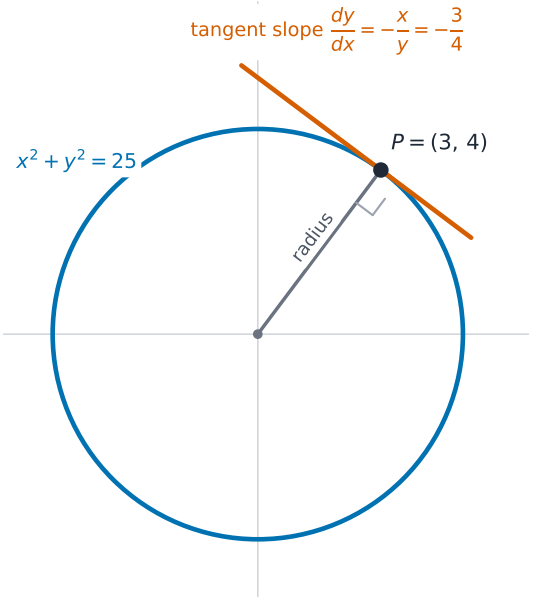
- Rate: $\dfrac{d\theta}{dt} = -54ke^{-kt}$; set equal to -9:

$$e^{-kt} = \frac{9}{54 \times 0.521} = \frac{9}{28.13} \implies T = \frac{\ln(28.13/9)}{0.521} = 2.19 \text{ (2 d.p.)}$$

- The **initial value** 初值 is 75 °C and the **long-run floor** 长期下限 is 21 °C.
- Check: at $T = 2.19$, $\theta = 21 + 54e^{-0.521(2.19)} = 21 + 54(0.320) = 38.3$ °C, falling at $28.13 \times 0.320 = 9.0$ °C/min ✓. The model's floor is 21 °C (room temperature) – sensible for large t.

Differentiation: rules, implicit and models

Differentiate $\sin kx$, $\cos kx$, $\tan kx$ ($\to k\sec^2 kx$), e^{kx}, $\ln kx$ ($\to \frac{1}{x}$), plus sums and differences.



Implicit differentiation gives the tangent to a circle, perpendicular to the radius

Three rules extend the reach: the **product rule** 乘积法则, the **quotient rule** 商法则 and the **chain rule** 链式法则.

$$\text{product: } (uv)' = u'v + uv' \qquad \text{quotient: } \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \text{chain: } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\sec x \to \sec x \tan x$, $\operatorname{cosec} x \to -\operatorname{cosec} x \cot x$, $\cot x \to -\operatorname{cosec}^2 x$ follow from the quotient rule. **Implicit differentiation** 隐函数求导 uses $\dfrac{dy}{dx} = 1 \div \dfrac{dx}{dy}$ when x is locally invertible as a differentiable function of y and $dx/dy \neq 0$.

Worked example – product and chain together (from the specification's own examples). Differentiate $e^{-x^2}\sin 3x$ (adapted from Edexcel IAL WMA13 June 2025 Q7).

- Product rule with $u = e^{-x^2}$ (chain: $u' = -2xe^{-x^2}$) and $v = \sin 3x$ ($v' = 3\cos 3x$):

$$\frac{dy}{dx} = -2xe^{-x^2}\sin 3x + 3e^{-x^2}\cos 3x = e^{-x^2}\big(3\cos 3x - 2x\sin 3x\big)$$

- Check at $x = 0$: derivative $= 1 \times (3 - 0) = 3$, and near 0 the curve behaves like $\sin 3x$ with gradient 3 ✓.

Integration: standard forms and recognition

Integrate the standard forms directly, and use **recognition forms** 识别积分 (spotting f' inside the integral):

$$\int \frac{\overbrace{6x}^{f'(x)}}{\underbrace{3x^2+4}_{f(x)}}\,dx = \ln|3x^2+4| + c$$

numerator a multiple of the inner derivative? then check by differentiating

$$\int x\,(3x^2+4)^4\,dx = \frac{(3x^2+4)^5}{30} + c$$

Spot f prime beside f

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + c \quad (n \neq -1)$$

Trig identities convert $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ into integrable forms ($\cos 2A \equiv 1 - 2\sin^2 A$, $\cos 2A \equiv 2\cos^2 A - 1$; $\tan^2 \equiv \sec^2 - 1$).

Worked example – identity first (adapted from Edexcel IAL WMA13 June 2025 Q5). Find $\int \sin^2 3x\,dx$.

- **Double angle** 二倍角 with $A = 3x$: $\cos 6x \equiv 1 - 2\sin^2 3x$, so $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$:

$$\int \sin^2 3x\,dx = \tfrac{1}{2}\int (1 - \cos 6x)\,dx = \tfrac{1}{2}\left(x - \tfrac{1}{6}\sin 6x\right) + c = \frac{x}{2} - \frac{\sin 6x}{12} + c$$

- Check by differentiating: $\frac{1}{2} - \frac{6\cos 6x}{12} = \frac{1}{2}(1 - \cos 6x) = \sin^2 3x$ ✓.

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Worked example – recognition. Find $\displaystyle\int \frac{6x}{3x^2+4}\,dx$.

- The numerator is the derivative of the denominator ($f' = 6x$, $f = 3x^2+4$):

$$\int \frac{6x}{3x^2+4}\,dx = \ln|3x^2+4| + c$$

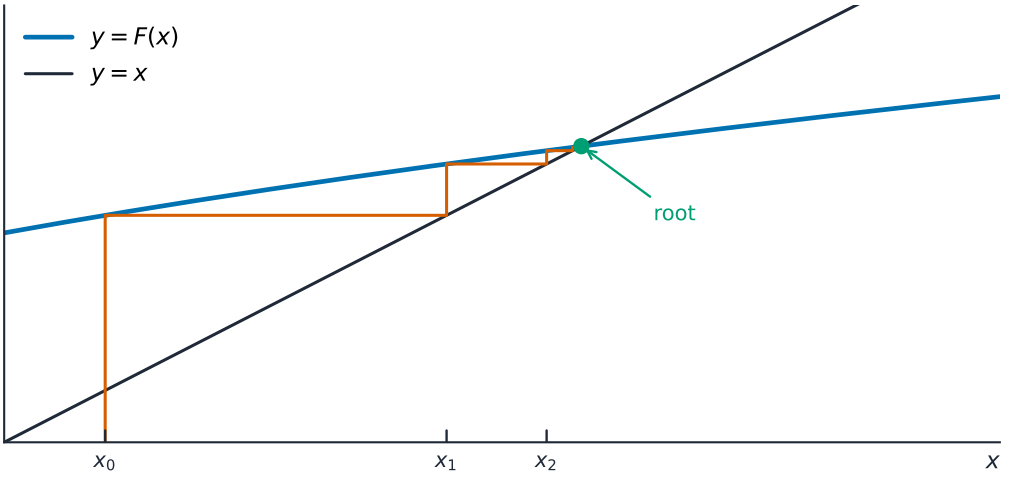
- Check: $\dfrac{d}{dx}\ln(3x^2+4) = \dfrac{6x}{3x^2+4}$ ✓. The powered form works the same way: with $u = 3x^2+4$ and $du = 6x\,dx$,

$$\int x\,(3x^2+4)^4\,dx = \tfrac{1}{6}\int u^4\,du = \frac{(3x^2+4)^5}{30} + c$$

and differentiating $\dfrac{(3x^2+4)^5}{30}$ gives $\dfrac{5(3x^2+4)^4\cdot 6x}{30} = x(3x^2+4)^4$ ✓.

Numerical methods: sign changes and iteration

- **Sign change** 变号: if f is continuous on $[a,b]$ and $f(a)f(b) < 0$, a root lies between a and b. Halve the interval to pin a root to any accuracy.
- **Iteration** 迭代: a rearrangement $x_{n+1} = f(x_n)$ from a start value converges to a **fixed point** 不动点 (or not – always check against a sign-change bound).



$x_{n+1} = F(x_n)$ · converges near the root if $|F'(x)| < 1$ there

Each step goes up to $y = F(x)$ then across to $y = x$; the staircase closes in on the root.

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Worked example – locate and iterate (adapted from Edexcel IAL WMA13 June 2025 Q7). The stationary point of $y = e^{-x^2}\sin 3x$ near $x = 0.4$ satisfies $x = \frac{1}{3}\arctan\frac{3}{2x}$.

(a) With $x_1 = 0.4$:

$$x_2 = \tfrac{1}{3}\arctan\left(\tfrac{3}{0.8}\right) = \tfrac{1}{3}\arctan(3.75) = 0.4367 \qquad x_4 = 0.4307 \text{ (4 d.p.)}$$

- Check the iteration converges: $|x_3 - x_2|$ small and the values settle toward 0.43 ✓.

(b) Show the root is 0.430 to 3 decimal places: define $g(x) = x - \frac{1}{3}\arctan\frac{3}{2x}$ (or the derivative expression from the worked example above):

$$g(0.4295) = -1.1 \times 10^{-3} < 0 \qquad g(0.4305) = +6.4 \times 10^{-5} > 0$$

- g is continuous, and the sign change between 0.4295 and 0.4305 brackets a root, so the root is 0.430 to 3 d.p. ✓.

Checklist before you call this unit done

- Simplify rational expressions; factorise before cancelling; divide by linear or quadratic divisors.
- Work with domain, range, composition and inverses; verify inverses by composing to x.
- Sketch and solve modulus equations by cases with checks; combine two transformations on one graph.
- Use sec, cosec, cot and their identities; know the inverse-function restricted domains.
- Convert $a\cos\theta + b\sin\theta$ to $R\cos(\theta \pm \alpha)$; read off maxima and solve.
- Prove compound and double angle identities; use them forwards and backwards.
- Solve $e^{ax+b} = p$ and $\ln(ax+b) = q$; recover $y = ax^n$ and $y = kb^x$ from log plots.
- Differentiate with product, quotient and chain rules, including sec / cosec / cot; use $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$.
- Model growth and decay; interpret the initial value and long-run floor; critique the model.
- Integrate standard forms; spot $\int f'/f$ and $\int f'(f)^n$; use identities on squared trig functions.
- Locate roots by sign change with a continuity statement; run and check an iteration.

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4 · Pure Mathematics 4

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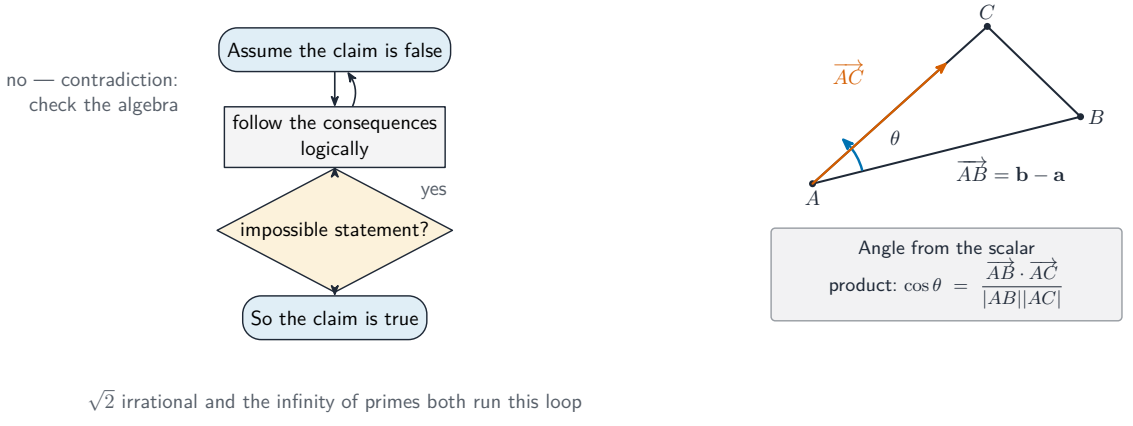
Pure Mathematics 4: proof, parametrics and the full calculus

Pure Mathematics 4 (P4) is the final compulsory pure unit of International A Level Mathematics and International A Level Pure Mathematics. P1–P3 are assumed and may be tested on the WMA14/01 paper (1 hour 30 minutes, 75 marks; same calculator and formula-booklet rules). This reference covers proof by contradiction, partial fractions, parametric equations, the general binomial series, implicit and parametric differentiation, differential equations, the extended integration methods, volumes of revolution, and vectors.

Must-know for P4 (not in the booklet):

Family	Formulae
binomial series	$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots$, valid $\lvert x\rvert < 1$ for any rational n
volume of revolution	$V = \pi \displaystyle\int_a^b y^2\,dx$ about the x-axis (parametric: $\pi \int y^2 \dfrac{dx}{dt}\,dt$)
vectors	column form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$; unit vector $\dfrac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$; $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Proof by contradiction



Left: proof by contradiction as a flow from assumption to impossibility to conclusion. Right: triangle ABC with position vectors and the angle from the scalar product.

Proof by contradiction 反证法 assumes the opposite of the claim, derives an im-

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possibility, and concludes the original claim. Two classics must be known:

Worked example – $\sqrt{2}$ is irrational. (A number is **irrational** 无理数 when no fraction $\frac{p}{q}$ equals it.) Suppose instead $\sqrt{2} = \frac{p}{q}$ in lowest terms (p, q coprime integers, $q \neq 0$).

- Squaring: $2 = \frac{p^2}{q^2} \implies p^2 = 2q^2$, so p^2 is even, so p is even: write $p = 2r$.
- Then $4r^2 = 2q^2 \implies q^2 = 2r^2$, so q is even too.
- Both p and q even contradicts $\frac{p}{q}$ being in lowest terms. ■

Worked example – there are infinitely many primes. Suppose finitely many: $p_1, \ldots, p_k$.

- Consider $N = p_1 p_2 \cdots p_k + 1$. Dividing N by any p_i leaves remainder 1.
- So N is either prime itself or has a prime factor not in the list – either way a prime outside $\{p_1, \ldots, p_k\}$, contradicting the assumption. ■

Worked example – a trigonometric contradiction (adapted from Edexcel IAL WMA14 June 2025 Q10). Prove that $|\sin x + \cos x| < 1$ for all $90° < x < 180°$.

- Suppose not: some x in the range has $|\sin x + \cos x| \geq 1$. Squaring (both sides non-negative):

$$(\sin x + \cos x)^2 \geq 1 \implies \sin^2 x + 2\sin x \cos x + \cos^2 x \geq 1 \implies 1 + \sin 2x \geq 1 \implies \sin 2x \geq 0$$

- But $90° < x < 180°$ gives $180° < 2x < 360°$, where $\sin 2x < 0$ – a contradiction.
- So no such x exists and $|\sin x + \cos x| < 1$ throughout the range. ■
- Check at $x = 135°$: $|\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}| = 0 < 1$ ✓; at $x = 100°$: $|0.985 - 0.174| = 0.811 < 1$ ✓.

Partial fractions

Decompose 分解 rational functions into **partial fractions** 部分分式. Denominators stay at repeated linear complexity: $(ax+b)(cx+d)(ex+f)$ or $(ax+b)(cx+d)^2$ with the **repeated factor** 重因式 $(cx+d)$ split across powers; quadratic factors are not required. **Improper fractions** 假分式 (numerator degree ≥ denominator's) divide first.

Worked example – distinct factors. Decompose $\dfrac{5x+1}{(x+2)(x-1)}$ (from the P2 sheet family):

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

- Check at $x = 0$: LHS $= \frac{1}{-2} = -\frac{1}{2}$; RHS $= \frac{3}{2} - 2 = -\frac{1}{2}$ ✓.

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

multiply through: $5x+1 = A(x-1) + B(x+2)$

put $x = 1$: $5(1)+1 = B(1+2)$
$\Rightarrow B = 2$

put $x = -2$: $5(-2)+1 = A(-2-1)$
$\Rightarrow A = 3$

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$
check: $3(x-1) + 2(x+2) = 5x+1$

Choose x to make one bracket zero: each substitution gives one constant

Worked example – repeated factor. Decompose $\dfrac{3x^2+7x+6}{x(x+1)^2}$.

- Set up with the repeated factor split across both powers:

$$\frac{3x^2+7x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

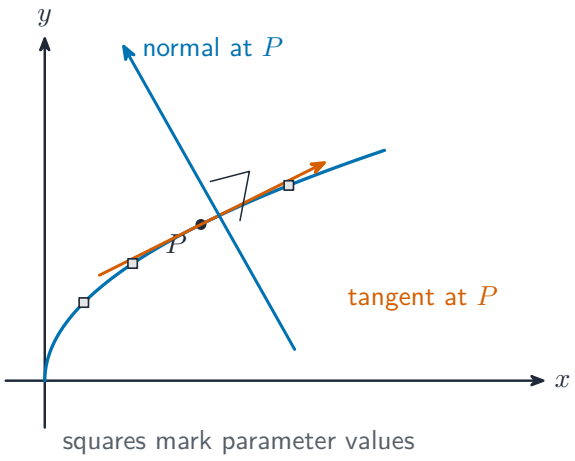
- Multiply through: $3x^2+7x+6 = A(x+1)^2 + Bx(x+1) + Cx$.
- At $x = 0$: $6 = A$. At $x = -1$: $3-7+6 = 2 = -C \Rightarrow C = -2$.
- Compare x^2: $3 = A + B \Rightarrow B = -3$.

$$\frac{3x^2+7x+6}{x(x+1)^2} = \frac{6}{x} - \frac{3}{x+1} - \frac{2}{(x+1)^2}$$

- Check at $x = 1$: LHS $= \frac{16}{4} = 4$; RHS $= 6 - \frac{3}{2} - \frac{1}{2} = 4$ ✓.

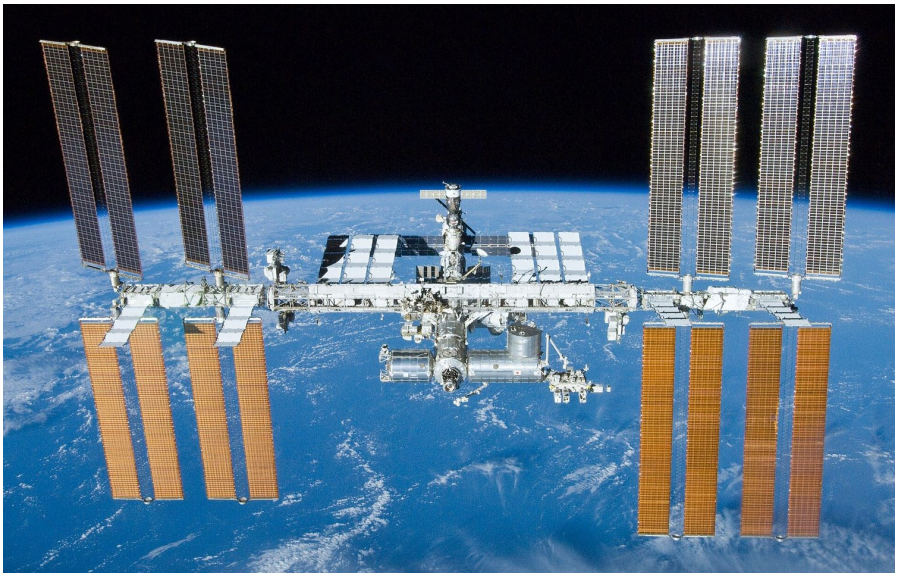
Parametric equations

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Parametric gradient: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$;
the normal takes its negative reciprocal.

A parametric curve with parameter ticks, the tangent at P using $dy/dx = (dy/dt)/(dx/dt)$, and the normal with the negative reciprocal slope.



The ISS in orbit: parametric equations give x and y separately as functions of time

Parametric equations 参数方程 give both coordinates through a parameter t: $x = f(t)$, $y = g(t)$. Convert to **Cartesian** 笛卡尔 form by making t the subject of one equation and substituting into the other (or by exploiting an identity). Never forget the excluded values of x where the parameter map breaks.

Worked example. A curve has $x = 2t$, $y = t^2 - 1$. Find the Cartesian equation.

- From $x = 2t$: $t = \dfrac{x}{2}$, so

$$y = \left(\frac{x}{2}\right)^2 - 1 = \frac{x^2}{4} - 1$$

- Check at $t = 3$: $(x, y) = (6, 8)$; Cartesian: $36/4 - 1 = 8$ ✓.

> **Worked example – parametric point and normal** (verified facts from Edexcel IAL WMA14 June 2025 Q5). A parametric curve passes through $P(-7, -3)$ at $t = 2$ with $\dfrac{dy}{dx} = -\frac{4}{5}$ there (from $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$).

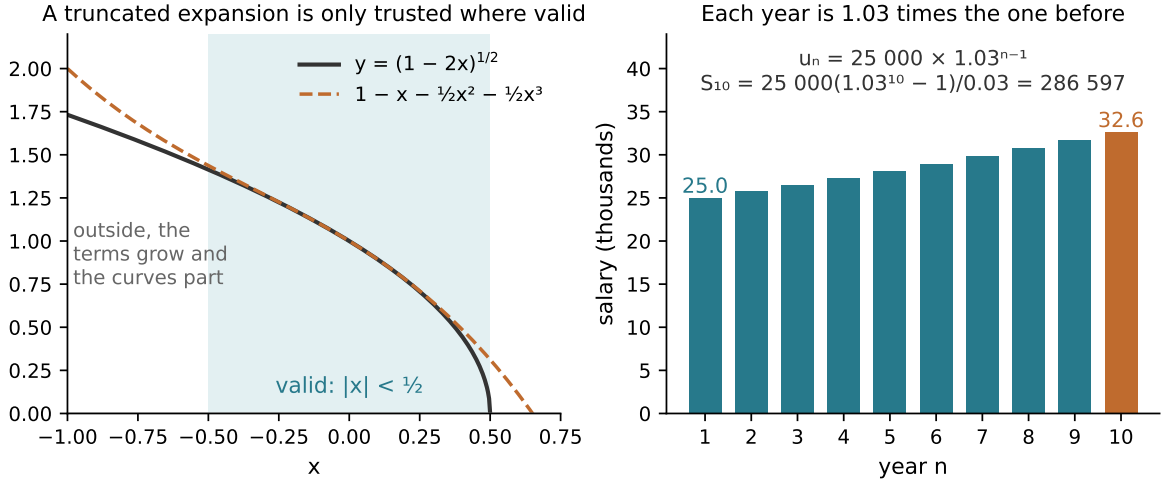
- Normal gradient: $-\dfrac{1}{-4/5} = \dfrac{5}{4}$.

$$y + 3 = \tfrac{5}{4}(x + 7) \implies 5x - 4y + 23 = 0$$

- Check: $5(-7) - 4(-3) + 23 = -35 + 12 + 23 = 0$ ✓.

The binomial series for any rational n

For any **rational** 有理数 n and $|x| < 1$:



The expansion matches the curve only where it is valid, $|x| < ½$; the salary bars grow by the same factor, 1.03, every year

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Factor first: $(a+bx)^n = a^n\left(1 + \frac{b}{a}x\right)^n$, valid for $|x| < \left|\frac{a}{b}\right|$. Decompose rational functions into partial fractions, then expand each piece. Powers do not terminate unless n is a positive integer.

> **Worked example** (adapted from Edexcel IAL WMA14 June 2025 Q8). Expand $\sqrt{4 + x}$ to three non-zero terms.

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- Factor: $\sqrt{4+x} = 2\left(1+\frac{x}{4}\right)^{1/2}$, valid $|x| < 4$, with $n = \frac{1}{2}$:

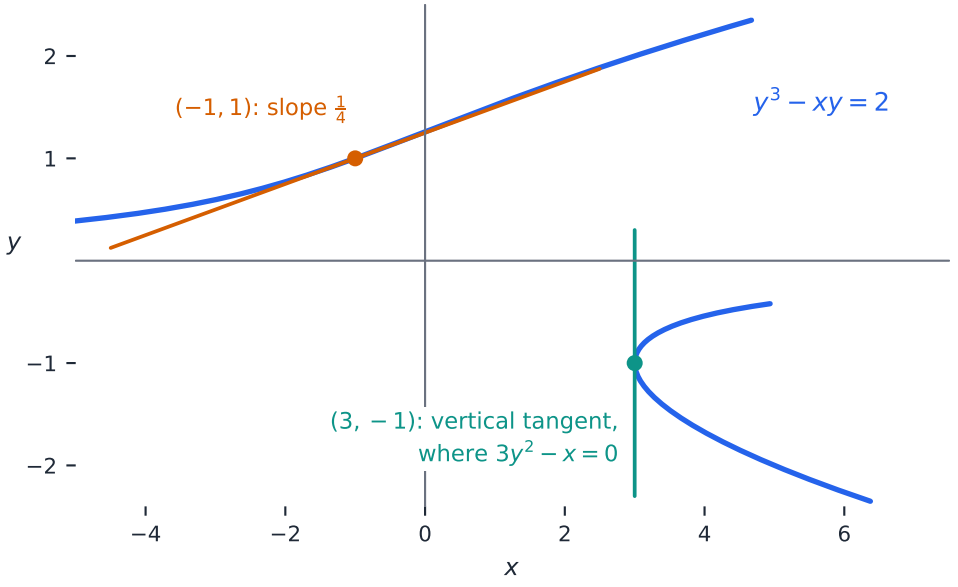
$$\left(1+\frac{x}{4}\right)^{1/2} = 1 + \frac{1}{2}\cdot\frac{x}{4} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x}{4}\right)^2 + \cdots = 1 + \frac{x}{8} - \frac{x^2}{128} + \cdots$$

$$\sqrt{4+x} = 2 + \frac{x}{4} - \frac{x^2}{64} + \cdots$$

- Check at $x = 1$: series gives $2 + 0.25 - 0.0156 = 2.234$; true $\sqrt{5} = 2.2361\ldots$ ✓ (the third term pulls the two-term value 2.25 down to within 0.002).

Implicit and parametric differentiation

Implicit differentiation 隐函数求导 differentiates both sides with respect to x, treating y as a function of x: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$, and products such as xy need the product rule: $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$. Then solve for $\frac{dy}{dx}$.



Where the denominator of $\frac{dy}{dx}$ is zero (and the numerator is not), the tangent is vertical

Parametric differentiation: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ when $dx/dt \neq 0$; otherwise check the tangent directly.

> **Worked example – an implicit tangent** (adapted from Edexcel IAL WMA14 June 2025 Q1). The curve $2y^2 - 6xy + 7e^{1-2x} = 27$ passes through the point $\left(\frac{1}{2},\ 4\right)$. Find the tangent there.

- Differentiate term by term:

$$4y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} - 14\mathrm{e}^{1-2x} = 0 \implies \frac{dy}{dx} = \frac{6y + 14\mathrm{e}^{1-2x}}{4y - 6x}$$

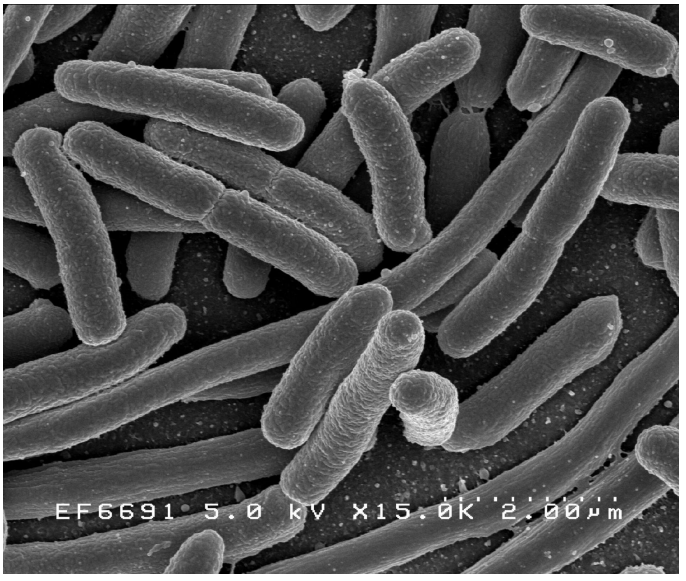
- At $\left(\frac{1}{2}, 4\right)$: $\mathrm{e}^{1-1} = 1$, so

$$\frac{dy}{dx} = \frac{24 + 14}{16 - 3} = \frac{38}{13}$$

- Tangent: $y - 4 = \frac{38}{13}\left(x - \frac{1}{2}\right) \implies 38x - 13y + 33 = 0$.
- Check: $38\left(\frac{1}{2}\right) - 13(4) + 33 = 19 - 52 + 33 = 0$ ✓.

Differential equations and connected rates

Form an equation from a rate description, then solve **separable** 可分离变量 equations:



E. coli under a microscope: differential equations model exponential growth of populations

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx$$

Check any value divided out while separating variables: constant solutions can be lost. Use a boundary condition for the **particular solution** 特解. **Connected rates of change** 连锁变化率 use the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$.

> **Worked example – a melting sphere** (Edexcel IAL WMA14 June 2025 Q2). A sphere of radius r cm has volume $V = \frac{4}{3}\pi r^3$ and melts with $\frac{dV}{dt} = k$ (constant, in cm^3 per second). Show $\frac{dr}{dt} \propto \frac{1}{r^2}$.

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- Differentiate the volume link: $\dfrac{dV}{dr} = 4\pi r^2$.
- Chain rule: $k = \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$, so
$$\frac{dr}{dt} = \frac{k}{4\pi r^2} = \frac{k}{4\pi} \cdot \frac{1}{r^2}$$
- The constant $\dfrac{k}{4\pi}$ multiplies $\dfrac{1}{r^2}$: the rate of decrease of the radius is inversely proportional to r^2. ■
- Interpretation: small spheres vanish disproportionately fast – the last millimetre melts quickest.

> **Worked example – a separable equation** (adapted from Edexcel IAL WMA14 June 2025 Q3). Solve $y\dfrac{dy}{dx} = 3\sec 2x \tan 2x$, given $y = 4$ at $x = \frac{\pi}{6}$.

- Separate: $\displaystyle\int y\,dy = \int 3\sec 2x \tan 2x\,dx$ (the right side integrates to $\frac{3}{2}\sec 2x$, reversing $\dfrac{d}{dx}\sec 2x = 2\sec 2x \tan 2x$):
$$\frac{y^2}{2} = \frac{3}{2}\sec 2x + c \implies y^2 = 3\sec 2x + C$$
- Substitute $\left(\frac{\pi}{6}, 4\right)$: $16 = 3\sec\frac{\pi}{3} + C = 3(2) + C$, so $C = 10$:
$$y^2 = 3\sec 2x + 10$$
- Check: $3(2) + 10 = 16 = 4^2$ ✓.

Integration: substitution, parts and partial fractions

- **Substitution** 换元积分 (usually given): rewrite the integral in u, adjust dx via $\dfrac{du}{dx}$, integrate, substitute back. It is the chain rule reversed.
- **Integration by parts** 分部积分: $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ – the product rule reversed. Know $\displaystyle\int \ln x\,dx = x\ln x - x + c$ and repeated parts for $\int x^2 \mathrm{e}^x dx$-type integrals.
- **Partial fractions** turn a rational integrand into sums of $\dfrac{A}{ax+b}$ (integrating to logs) and $\dfrac{C}{(ax+b)^2}$ (to powers).

> **Worked example – parts with a repeat in sight.** Find $\displaystyle\int \mathrm{e}^x \cos x\,dx$ (needed for volumes of revolution with trig curves).

- Parts with $u = \mathrm{e}^x$, $\frac{dv}{dx} = \cos x$:

$$\int \mathrm{e}^x \cos x \, dx = \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x \, dx$$

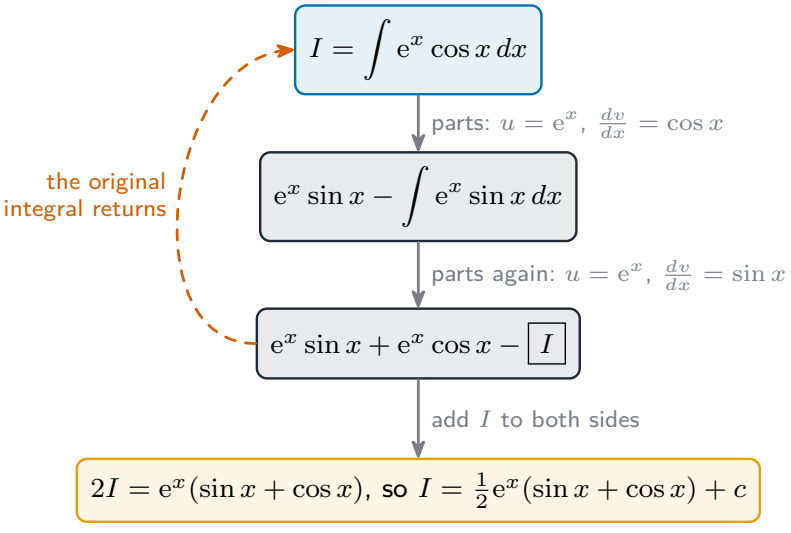
- Parts again on the new integral ($u = \mathrm{e}^x$, $\frac{dv}{dx} = \sin x$):

$$= \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x \, dx$$

- The original integral reappears; collect it:

$$2\int \mathrm{e}^x \cos x \, dx = \mathrm{e}^x(\sin x + \cos x) + c \implies \int \mathrm{e}^x \cos x \, dx = \tfrac{1}{2}\mathrm{e}^x(\sin x + \cos x) + c$$

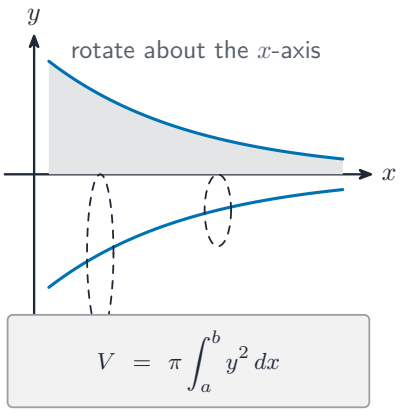
- Check by differentiating: $\frac{1}{2}\mathrm{e}^x(\sin x + \cos x) + \frac{1}{2}\mathrm{e}^x(\cos x - \sin x) = \mathrm{e}^x \cos x$ ✓.

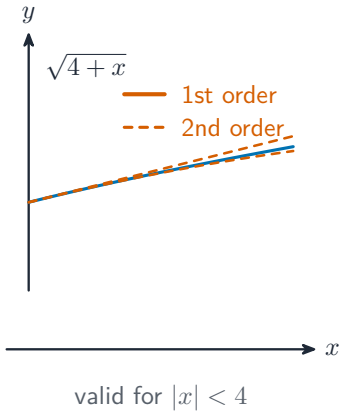


Two rounds of parts bring the original integral back. Collect it: 2I = e to the x (sin x + cos x)

Volumes of revolution and parametric areas

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Left: a region rotating about the x-axis with suggested cross-sections and $V = \pi\int y^2\,dx$. Right: binomial partial sums of $\sqrt{(4+x)}$ approaching the true curve.



A potter's vessel is a solid of revolution: rotate a curve about an axis to find its volume

Rotating the region under $y = f(x) \geq 0$ about the x-axis sweeps a solid whose volume is

the **volume of revolution** 旋转体体积

$$V = \pi \int_a^b y^2\,dx$$

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Square the function **before** integrating, use identities to handle $\sin^2/\cos^2$, and split by linearity. With parametric equations, replace $y^2\,dx$ by $y^2\dfrac{dx}{dt}\,dt$ between parameter values; areas under parametric curves use $y\dfrac{dx}{dt}\,dt$.

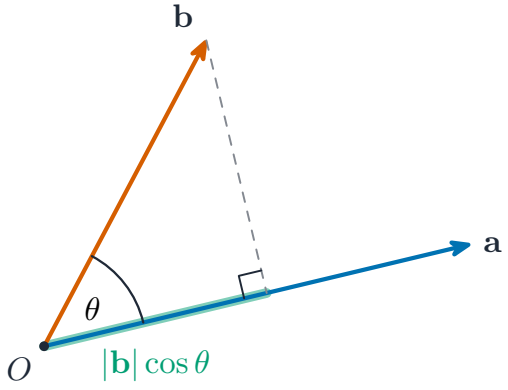
> **Worked example – volume with an identity** (structure adapted from Edexcel IAL WMA14 June 2025 Q9). The region under $y = 3\cos x$ for $0 \le x \le \frac{\pi}{2}$ rotates about the x-axis; find the exact volume.

$$V = \pi\int_0^{\pi/2} 9\cos^2 x\,dx = 9\pi\int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2x)\,dx = \frac{9\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{9\pi}{2}\cdot\frac{\pi}{2} = \frac{9\pi^2}{4}$$

- Check: the average of $\cos^2 x$ over a half-turn is $\frac{1}{2}$, so $V \approx 9\pi \times \frac{1}{2} \times \frac{\pi}{2} = \frac{9\pi^2}{4}$ ✓ exactly.

Vectors

A **vector** 向量 has magnitude and direction, written $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ or as a column.



The scalar product picks out $|\mathbf{b}|\cos\theta$, how far **b** *reaches along* **a**.

The **magnitude** 模 is $|\mathbf{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$; for $\mathbf{a} \neq \mathbf{0}$, the **unit vector** 单位向量 in its direction is $\mathbf{a}/|\mathbf{a}|$. The zero vector has no direction.

Addition and scalar multiplication have geometric meanings (translations, scaling, **parallel** 平行 when one is a multiple of the other).

The **scalar product** 数量积 $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ gives angles between nonzero vectors through $\cos\theta$. The **position vector** 位置向量 of B relative to A is $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$.

> **Worked example – a 45° triangle** (Edexcel IAL WMA14 June 2025 Q6, fully verified). $A = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $B = 5\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $C = 3\mathbf{i}+\alpha\mathbf{j}+5\mathbf{k}$, and angle $BAC = 45°$.

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- $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$, so $|\overrightarrow{AB}| = \sqrt{32}$; $\overrightarrow{AC} = \begin{pmatrix} 2 \\ \alpha - 2 \\ 8 \end{pmatrix}$.
- Scalar product: $\overrightarrow{AB} \cdot \overrightarrow{AC} = 8 + 0 + 32 = 40$.
- Angle condition with $\cos 45° = \frac{1}{\sqrt{2}}$:

$$\frac{40}{\sqrt{32}\sqrt{68 + (\alpha - 2)^2}} = \frac{1}{\sqrt{2}} \implies \sqrt{68 + (\alpha - 2)^2} = \frac{40\sqrt{2}}{\sqrt{32}} = \frac{40\sqrt{2}}{4\sqrt{2}} = 10$$

$$(\alpha - 2)^2 = 100 - 68 = 32 \implies \alpha = 2 \pm 4\sqrt{2}$$

- Check: $|\overrightarrow{AB}||\overrightarrow{AC}| \cos 45° = \sqrt{32} \times 10 \times \frac{1}{\sqrt{2}} = 4\sqrt{2} \times 10 \times \frac{1}{\sqrt{2}} = 40$ ✓ matches the scalar product.

Checklist before you call this unit done

- Structure contradiction proofs ($\sqrt{2}$, primes) with the assumption stated and the contradiction named.
- Decompose into partial fractions including repeated linear factors; divide improper fractions first; verify by a substituted value.
- Convert parametric $\leftrightarrow$ Cartesian with excluded values; find parametric tangents and normals via dy/dx = (dy/dt)/(dx/dt).
- Expand $(a + bx)^n$ for rational n with the validity condition; combine expansions of partial fractions.
- Differentiate implicitly (product rule on mixed terms) and parametrically; form differential equations from connected rates.
- Solve separable equations with a boundary condition; check the particular solution.
- Integrate by substitution, parts (including repeats, $\ln x$), and partial fractions.
- Compute volumes of revolution $\pi \int y^2 dx$ and parametric areas via dx/dt.
- Use vectors: magnitude, unit vectors, position vectors; scalar-product angles.

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Further Pure Mathematics 1

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 1: complex numbers, matrices and proof

Further Pure Mathematics 1 (FP1, paper WFM01/01, 1 hour 30 minutes, 75 marks) is compulsory for International AS and A Level Further Mathematics and for International AS and A Level Pure Mathematics. P1–P2 are assumed and may be tested, including sign-change location of roots, rotating shapes about the origin, and dividing a cubic or quartic by a quadratic. This reference covers complex numbers, roots of quadratic equations, numerical methods, the parabola and rectangular hyperbola, matrix algebra, transformations, finite series and proof by induction.

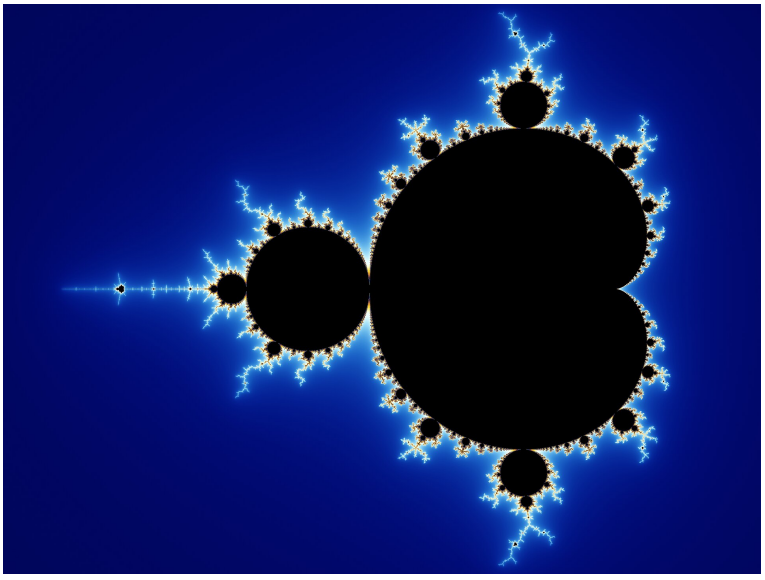
Must-know for FP1 (not in the booklet):

Family	Formulae
quadratic roots	for $ax^2+bx+c=0$ with roots α,β: $\alpha+\beta=-\dfrac{b}{a}, \quad \alpha\beta=\dfrac{c}{a}$
series	$\displaystyle\sum_{r=1}^{n} r=\tfrac{1}{2}n(n+1)$
complex numbers	$z=a+bi$, $\lvert z\rvert=\sqrt{a^2+b^2}$, $\bar{z}=a-bi$, $\lvert z_1z_2\rvert=\lvert z_1\rvert\lvert z_2\rvert$
conics	parabola $y^2=4ax$ (parametric $x=at^2$, $y=2at$); rectangular hyperbola $xy=c^2$ (parametric $x=ct$, $y=\frac{c}{t}$)
matrices	$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-bc, \quad \mathbf{M}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$, $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$

Complex numbers

A **complex number** 复数 is $z=a+bi$ with $i^2=-1$; a is the **real part** 实部, b the **imaginary part** 虚部. The **conjugate** 共轭复数 $\bar{z}=a-bi$ reverses the sign of the imaginary part, and $z\bar{z}=a^2+b^2=|z|^2$ – the trick that makes the denominator real: $\dfrac{z_1}{z_2}=\dfrac{z_1\bar{z}_2}{|z_2|^2}$. Two complex numbers are equal only when both real and imaginary parts match. The polar form $z=r(\cos\theta+i\sin\theta)$ pairs the **modulus** 模 r with the **argument** 辐角 (defined only when $z\neq 0$) θ; multiplying complex numbers multiplies their moduli ($|z_1z_2|=|z_1||z_2|$).

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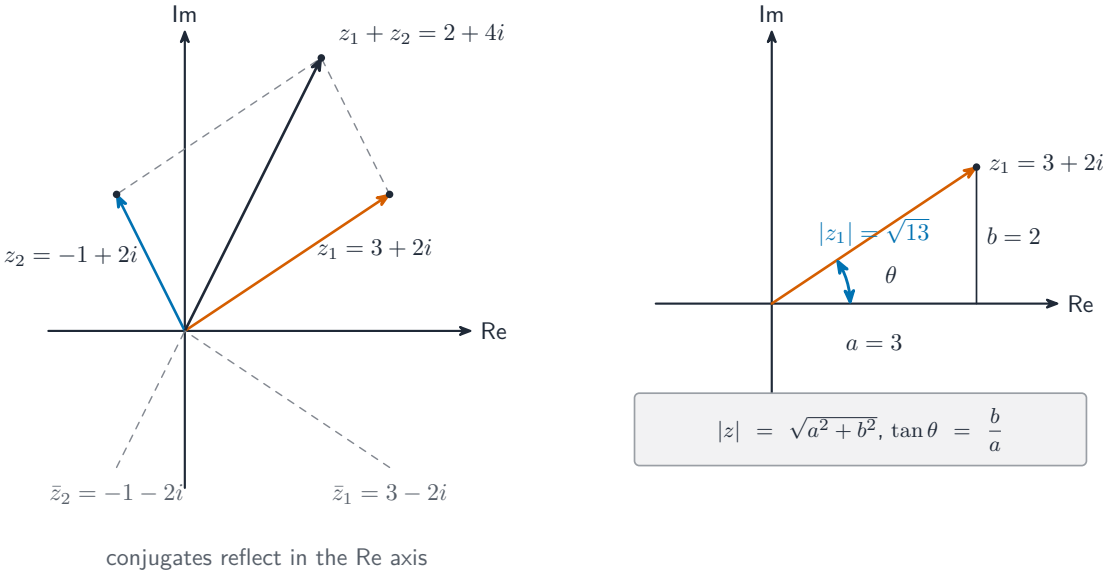


The Mandelbrot set: the complex numbers c for which repeating $z \to z^2 + c$, starting from $z = 0$, stays bounded

> **Worked example – a real sum pins the imaginary part.** $z_1 = r(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$ with $r > 0$, and z_2 has modulus 5. Since $|z_1 z_2| = |z_1||z_2| = 5r = 15$, we get $r = 3$. So $z_1 = 3(-\frac{\sqrt{3}}{2} - \frac{1}{2}i) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$. If $z_1 + z_2$ is real, its imaginary part is zero, so $z_2 = a + \frac{3}{2}i$; then $a^2 + \frac{9}{4} = 25$ gives $a = \pm\frac{\sqrt{91}}{2} \approx \pm 4.77$. Both $z_2 = \frac{\sqrt{91}}{2} + \frac{3}{2}i$ and its reflection in the imaginary axis work.

On an **Argand diagram** 阿甘特图 (Argand diagram) plot $z = a + bi$ as the point (a, b): addition is vector addition, the conjugate reflects in the real axis, and $|z|$ is the distance from the origin.

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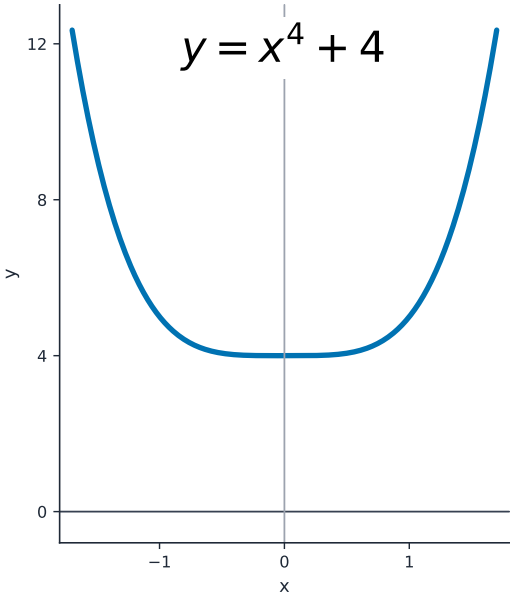


Left: an Argand diagram showing z1 = 3 + 2i, z2 = −1 + 2i, their sum 2 + 4i as a vector parallelogram, and both conjugates dashed below the real axis. Right: the same points with modulus arrows and the argument of z1 marked from the positive real axis.

Complex roots of polynomials

For a quadratic with real coefficients and negative discriminant, complete the square: $x^2 - 6x + 13 = (x - 3)^2 + 4 = 0$ gives $x = 3 \pm 2i$. The roots are a conjugate pair – and this survives degree: **if $f(z_1) = 0$ for real coefficients, then $f(\bar{z}_1) = 0$ too**. So a real cubic has either three real roots or one real root plus one conjugate pair, and $(z - z_1)(z - \bar{z}_1) = z^2 - 2\,\mathrm{Re}(z_1)z + |z_1|^2$ is a real quadratic factor. Divide it out to reach the rest.

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No x-intercepts

Four complex roots

$$1+i, \quad 1-i$$

$$-1+i, \quad -1-i$$

Sum = 0

Product = 4

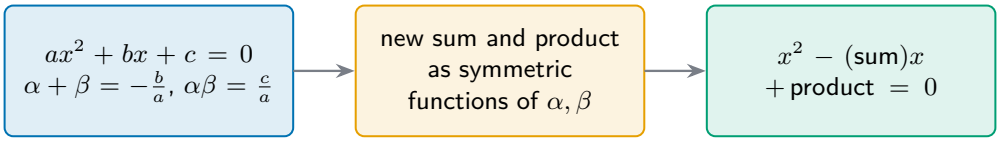
The four roots of $x^4 + 4 = 0$ are $1+i, 1-i, -1+i, -1-i$. Their sum is 0 and their product is 4, although the real graph has no x-intercepts.

> **Worked example – a quartic with a given complex root (WFM01 June 2025 Q7).** $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3+5i$, so $3-5i$ is also a root and
>
> $$z^2 - 6z + 34 \text{ is a factor} \quad (\text{sum } 6,\ \text{product } 9+25=34).$$
>
> Writing $f(z) = (z^2 - 6z + 34)(Pz^2 + bz + c)$ and comparing: $34c = 68$ so $c = 2$; $-6c + 34b = 192$ so $34b = 204$, $b = 6$; then $b - 6P = -36$ gives $P = 7$ and $c - 6b + 34P = Q$ gives $Q = 204$. (Check against $f(3+5i) = 0$: the real equation $644P + 16Q = 7772$ holds, since $644(7) + 16(204) = 4508 + 3264 = 7772$.) The other roots solve $7z^2 + 6z + 2 = 0$:
>
> $$z = \frac{-6 \pm \sqrt{36 - 56}}{14} = \frac{-6 \pm 2i\sqrt{5}}{14} = \frac{-3 \pm i\sqrt{5}}{7}.$$
>
> Without a calculator: only $b^2 - 4ac < 0$ arithmetic and the conjugate fact are used.

Roots of quadratic equations

For $ax^2 + bx + c = 0$ with roots α, β: $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Symmetric expressions in α, β then need no individual roots: $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$, $\alpha^3 + \beta^3 \equiv (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$, and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$. To build a new quadratic, express the new **sum** and **product** in terms of the old ones; the equation is $x^2 - (\text{sum})x + \text{product} = 0$.

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$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

no individual root is ever needed

New roots without solving

Worked example – new roots without solving (WFM01 June 2025 Q6). $f(x) = 3x^2 + kx - 5$ has roots α, β, so $\alpha\beta = -\frac{5}{3}$. If $\alpha + \beta = 9\alpha\beta$, then $-\frac{k}{3} = -15$, so $k = 45$ and $\alpha + \beta = -15$. For roots $\alpha^2 + \beta$ and $\alpha + \beta^2$:

$$\text{sum} = \alpha^2 + \beta^2 + \alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta + (\alpha + \beta) = 225 + \tfrac{10}{3} - 15 = \tfrac{640}{3},$$

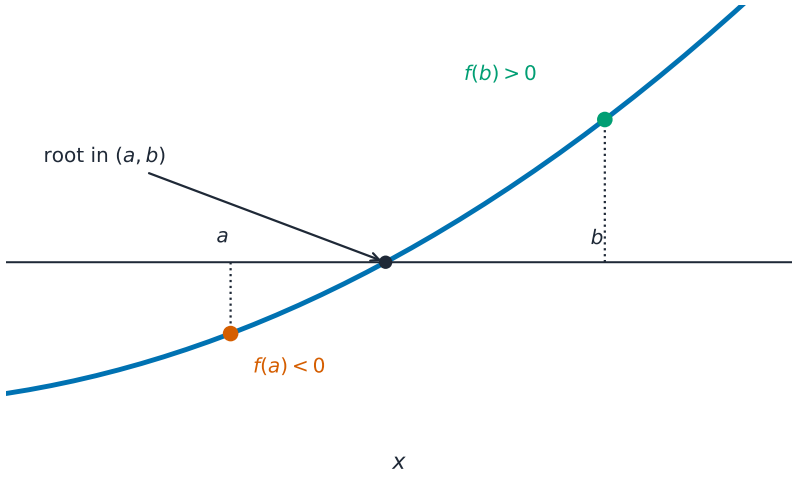
$$\text{product} = (\alpha^3 + \beta^3) + \alpha\beta + (\alpha\beta)^2 = -3450 - \tfrac{5}{3} + \tfrac{25}{9} = -\tfrac{31040}{9}.$$

The new equation is $9x^2 - 1920x - 31040 = 0$ (multiply $x^2 - \frac{640}{3}x - \frac{31040}{9} = 0$ by 9).

Numerical solution of equations

Three iterative locators: bisection and interpolation need a continuous function on their bracket; Newton–Raphson also needs a defined, nonzero derivative at each step and can fail to converge. **interval bisection** 区间二分法 halves an interval carrying a sign change; **linear interpolation** 线性插值 chords from $(a, f(a))$ to $(b, f(b))$ and takes the chord's zero $\beta \approx a - \dfrac{(b-a)f(a)}{f(b) - f(a)}$; the **Newton–Raphson method** 牛顿-拉夫逊法 (formula given in the booklet) iterates $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Show every evaluation – "calculator-only" answers score nothing.

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if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

For a continuous function on $[a, b]$, opposite endpoint signs trap a root between a and b.

> **Worked example – bisection and interpolation (WFM01 June 2025 Q3).** (i) $f(x) = x^2 + 5 - 8^{5x}$ has a single root α in $[0, 1]$: $f(0) = 4 > 0$, $f(0.5) = 5.25 - 8^{2.5} = -175.77 < 0$, so halve: $f(0.25) = 5.0625 - 8^{1.25} = -8.392 < 0$. The sign change sits in $[0, 0.25]$. (ii) $g(x) = 3^{\sin x} - 3\cos x$ (radians): $g(4) = 2.396 > 0$ and $g(5) = -0.502 < 0$, and g is continuous, so a root β lies in $[4, 5]$. Interpolating once: $\beta \approx 4 + \dfrac{2.396}{2.396 + 0.502} = 4.827$ (4 s.f.).

Newton–Raphson (original example). Solve $x^3 - 2x - 5 = 0$ from $x_0 = 2$: $f(2) = -1$, $f'(x) = 3x^2 - 2$ so $f'(2) = 10$ and $x_1 = 2 - \frac{-1}{10} = 2.1$; then $f(2.1) = 0.061$, $f'(2.1) = 11.23$, $x_2 = 2.0946$. The iterates approach the root, approximately 2.09455 (check: $2.0946^3 - 2(2.0946) - 5 \approx +0.0005$).

Parabola and rectangular hyperbola

The **parabola** 抛物线 $y^2 = 4ax$ has parametric form $x = at^2$, $y = 2at$; the **rectangular hyperbola** 直角双曲线 $xy = c^2$ has $x = ct$, $y = \frac{c}{t}$. Tangents and normals come from differentiating the Cartesian equation (parametric differentiation is not required): $y^2 = 4ax$ gives $\frac{dy}{dx} = \frac{2a}{y} = \frac{1}{t}$ at parameter t, and $xy = c^2$ gives $y = \frac{c^2}{x}$ so $\frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{1}{t^2}$ at $x = ct$.

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A suspension bridge: its main cable hangs close to a parabola, a curve engineers really build

Worked example – normal to a hyperbola (WFM01 June 2025 Q4). For $xy = 81$ at $P(9t, \frac{9}{t})$: $\frac{dy}{dx} = -\frac{81}{x^2} = -\frac{1}{t^2}$, so the normal has gradient t^2:

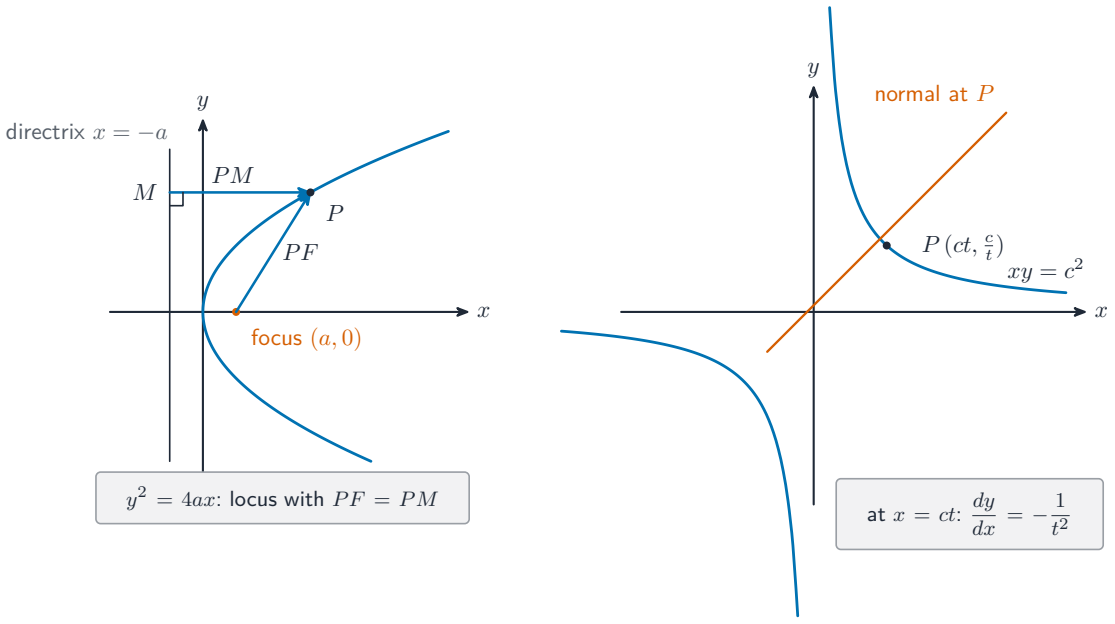
$$y - \tfrac{9}{t} = t^2(x - 9t) \implies ty = t^3x + 9(1 - t^4).$$

At the y-axis, $A = (0, \frac{9(1-t^4)}{t})$. With $t = \frac{1}{3}$: $P = (3, 27)$ and $A = (0, \frac{80}{3})$, so triangle OPA has area $\frac{1}{2} \times 3 \times \frac{80}{3} = 40$.

The parabola is the locus of points equidistant from the **focus** 焦点 $(a, 0)$ and the **directrix** 准线 $x = -a$. Distances, not algebra, unlock the geometry questions.

Worked example – focus-directrix distances (WFM01 June 2025 Q9). P lies on the parabola 5 units from the focus, so P is also 5 units from the directrix ($PF = PM = 5$, with PM perpendicular to the directrix). Points Q, R on the directrix satisfy $PQ = PR = 10$. Each right triangle PQM gives $QM = \sqrt{10^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$, and M lies between Q and R, so $QR = 10\sqrt{3}$.

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Left: the parabola $y^2 = 4ax$ with focus $(a, 0)$, directrix $x = -a$, and a point P whose equal perpendicular distance to focus and directrix ($PF = PM$) is marked with ticks. Right: the rectangular hyperbola $xy = c^2$ with the parametric point $(ct, c/t)$ and the normal line at that point.

Matrix algebra

Matrices add entry-wise (same shape), scale entry-wise, and multiply row-by-column ($(\mathbf{AB})_{ik} = \sum_j A_{ij}B_{jk}$; $\mathbf{AB} \neq \mathbf{BA}$ in general). For 2×2 matrices $\det \mathbf{M} = ad - bc$: zero means **singular** 奇异 (no inverse); otherwise $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$. Inverses reverse order: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ – undo the last transformation first.

> **Worked example – an inverse that must stay consistent (WFM01 June 2025 Q1).** $\mathbf{M} = \begin{pmatrix} 1 & a \\ 3 & -5 \end{pmatrix}$ with $a \neq -\frac{5}{3}$:
>
> $$\mathbf{M}^{-1} = \frac{1}{-(5+3a)}\begin{pmatrix} -5 & -a \\ -3 & 1 \end{pmatrix}.$$
>
> Given $\mathbf{M}^{-1} = 2\mathbf{M} + 8\mathbf{I} = \begin{pmatrix} 10 & 2a \\ 6 & -2 \end{pmatrix}$, compare entries: $\dfrac{-3}{-(5+3a)} = 6$ gives $5 + 3a = \frac{1}{2}$, and every other entry confirms it, so $a = -\frac{3}{2}$.

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Transformations using matrices

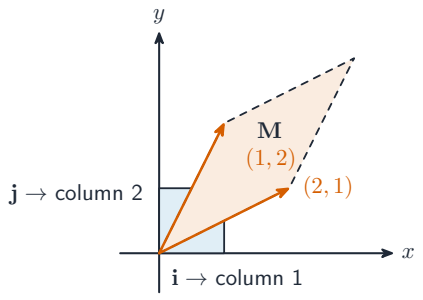
A 2×2 matrix $\mathbf{M}$ transforms column vectors by $\mathbf{y}=\mathbf{Mx}$; its **columns are the images of i and j** – read any transformation straight off the matrix. The standard results:

Transformation	Matrix
reflection in the x-axis	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
reflection in the y-axis	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$
reflection in $y=x$ / $y=-x$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ / $\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$
rotation, angle θ anticlockwise about O	$\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$
stretch, factor k parallel to the x-axis	$\begin{pmatrix}k&0\\0&1\end{pmatrix}$ (swap for the y-axis)
enlargement, scale factor k about O	$\begin{pmatrix}k&0\\0&k\end{pmatrix}$

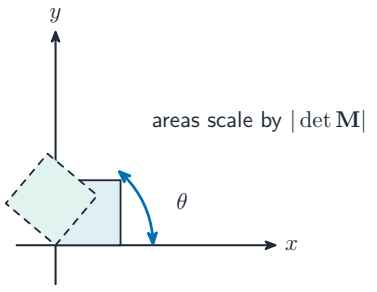
For combinations, **AB** means "apply **B** first, then **A**". The inverse transformation undoes it, and $|\det\mathbf{M}|$ is the **area scale factor** 面积比例因子: areas multiply by the determinant (negative determinants add a reflection).

> **Worked example – describing and combining (WFM01 June 2025 Q2).** $\mathbf{P}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ maps $\mathbf{i}\to(0,-1)^T$ and $\mathbf{j}\to(1,0)^T$: a quarter-turn clockwise (270° anticlockwise) about O. $\mathbf{Q}=5\mathbf{I}$ is an enlargement, scale factor 5, centre O. Reflection in $y=-x$ is $\mathbf{R}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$, so $A=\mathbf{R}(4,3)^T=(-3,-4)$. If $\mathbf{Q}$ follows $\mathbf{P}$ and maps B to A, then $B=\mathbf{P}^{-1}\mathbf{Q}^{-1}A=\mathbf{P}^{-1}\cdot\frac{1}{5}(-3,-4)^T=\frac{1}{5}(4,-3)^T=(0.8,-0.6)$ – $\mathbf{P}$'s own inverse is its transpose, so this is arithmetic only.

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$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: columns are the images of $\mathbf{i}, \mathbf{j}$



rotation by θ: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Left: the unit square with basis vectors i and j, and its image under the matrix with columns equal to the images of i and j. Right: the same transformation shown as a rotation and stretch combination, with the determinant marked as the area scale factor between the two parallelograms.

Series

With $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (and $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$ from the booklet), any polynomial summand splits into standard pieces – including sums to an even limit $2n$ (substitute $k = 2n$ into the standard formulae) and shifted sums like $\sum(r+1)^2$.

> **Worked example – a sum to 2n (WFM01 June 2025 Q8a).** Show $\sum_{r=1}^{2n}(2r^2 - 1) = \frac{4}{3}n(n+1)(an+b)$:

$$2\sum_{1}^{2n} r^2 - 2n = 2 \cdot \frac{(2n)(2n+1)(4n+1)}{6} - 2n = \frac{2n(2n+1)(4n+1) - 6n}{3} = \frac{2n(8n^2+6n-2)}{3} = \frac{4n(4n^2+3n-1)}{3},$$

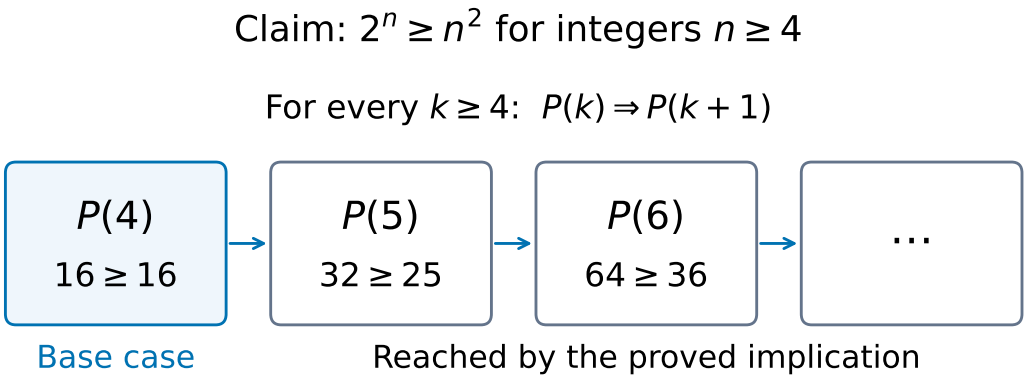
and $4n^2 + 3n - 1 = (n+1)(4n-1)$, so the sum is $\frac{4}{3}n(n+1)(4n-1)$: $a = 4$, $b = -1$. (Check $n = 1$: $1 + 7 = 8$ and $\frac{4}{3} \cdot 1 \cdot 2 \cdot 3 = 8$.)

Proof by mathematical induction

Induction 数学归纳法 proves a statement $P(n)$ for all positive integers in three moves: verify $P(1)$; assume $P(k)$; deduce $P(k+1)$ from it – then close with "true for $n = 1$, and truth for $n = k$ implies truth for $n = k + 1$, so by induction true for all positive integers n". The closing sentence is part of the proof. Four assessed families: summation formulae, divisibility, sequences given by a recurrence, and matrix powers.



Falling dominoes: prove the first case, prove each case knocks over the next, and every case falls



Worked example – a summation (WFM01 June 2025 Q8b). Prove $\sum_{r=1}^{n} r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$. For $n=1$: LHS $= 1\cdot 1^2 = 1$ and RHS $= \frac{1}{4}\cdot 1\cdot 2\cdot 2 = 1$ – true. Assume true for $n=k$; the $(k+1)$th term is $(k+1)(3k+1)^2$, so

$$\sum_{r=1}^{k+1} r(3r-2)^2 = \tfrac{1}{4}k^2(k+1)(9k-7) + (k+1)(3k+1)^2 = \tfrac{1}{4}(k+1)\big[k^2(9k-7) + 4(3k+1)^2\big].$$

Expand the bracket: $9k^3 - 7k^2 + 36k^2 + 24k + 4 = 9k^3 + 29k^2 + 24k + 4$. The target for $n = k+1$ is $\frac{1}{4}(k+1)^2(k+2)(9k+2)$, and its bracket $(k+1)(k+2)(9k+2)$ expands as $(k^2+3k+2)(9k+2) = 9k^3 + 29k^2 + 24k + 4$ – identical. So the statement holds for $n = k+1$, and by induction for all positive integers n.

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Divisibility (original example). Prove $n^3 + 2n$ is divisible by 3. For $n = 1$: 3. Assume $k^3+2k = 3m$; then $(k+1)^3+2(k+1) = (k^3+2k)+3k^2+3k+3 = 3(m+k^2+k+1)$, still a multiple of 3. By induction, true for all n.

Matrix powers (original example). Prove $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$. True for $n = 1$. If true for k, then $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & k+1\\ 0 & 1\end{pmatrix}$, completing the induction.

Checklist before you call this unit done

- Work with $a + bi$ and $r(\cos\theta + i\sin\theta)$: modulus, argument, conjugate; quotients made real.
- Use conjugate pairs to factor cubics and quartics with real coefficients; find all roots without a calculator.
- Convert symmetric expressions via $\alpha + \beta$ and $\alpha\beta$; form new quadratics for transformed roots.
- Locate roots by bisection and linear interpolation with every evaluation shown; iterate Newton–Raphson from a given start.
- Use $y^2 = 4ax$ and $xy = c^2$ parametrics; tangents/normals via Cartesian differentiation; focus-directrix distances.
- Invert 2×2 matrices (and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$); solve matrix equations entry by entry.
- Identify reflections, rotations, stretches, enlargements from matrices; combine in the right order; use $|\det|$ as the area scale factor.
- Sum series to n and $2n$ with the standard formulae; prove series, divisibility, recurrence and matrix results by induction with a closing statement.

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Further Pure Mathematics 2

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 2: inequalities, further complex numbers and differential equations

Further Pure Mathematics 2 (FP2, paper WFM02/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) builds directly on P1–P4 and FP1 and counts one sixth of an International Advanced Level. Every unit is equally weighted, so FP2 material can also surface inside Further Mathematics. P1–P4 assumed content stays live: partial fractions, parametric and implicit differentiation, integration by parts and substitution, and the whole FP1 complex-number and proof toolkit. This reference covers modulus inequalities, the method of differences, Euler's relation and de Moivre's theorem, loci and z→w transformations, first and second order differential equations, Maclaurin and Taylor series, and polar coordinates.

Must-know for FP2 (assumed, not re-taught):

Family	Formulae
modulus	$\lvert x\rvert = x$ for $x \geq 0$ and $-x$ for $x < 0$; $\lvert z\rvert = \sqrt{a^2+b^2}$, $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
sums	$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum r^2$, $\sum r^3$ (booklet)
algebra	partial fractions, e.g. $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$ (P4)
calculus	product, quotient and chain rules; integration by substitution, parts, partial fractions (P4)
differential equations	separation of variables; exponential growth/decay $y = Ae^{kt}$ (P4)

Inequalities involving the modulus sign

Two safe moves and one banned move. Safe: split into cases from the definition of the modulus ($\lvert f\rvert = f$ where $f \geq 0$, $-f$ where $f < 0$), and square both sides of $\lvert A\rvert > \lvert B\rvert$ to get $A^2 > B^2$. Banned: multiplying or dividing by an expression whose sign you have not fixed – case-split on that sign instead, and intersect each case with its own domain.

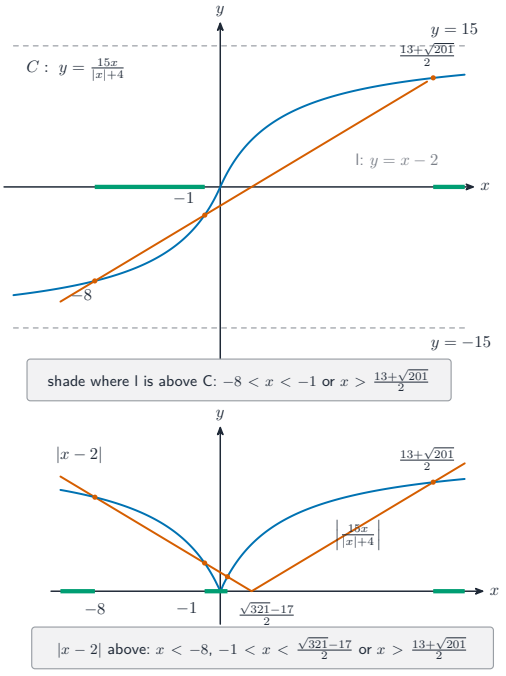
> **Worked example – one branch dies (specimen style).** Solve $\lvert x^2-1\rvert > 2(x+1)$. Where $x^2 \geq 1$: $x^2-2x-3 > 0$, so $(x-3)(x+1) > 0$, giving $x > 3$ or $x < -1$; keeping only $x \leq -1$ or $x \geq 1$ leaves $x < -1$ or $x > 3$. Where $x^2 < 1$: $1 - x^2 > 2x + 2$ becomes $(x+1)^2 < 0$ – impossible. The solution is $x < -1$ or $x > 3$. A dead branch is a result, not a mistake: report it.

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Worked example – modulus on both sides (WFM02 June 2025 Q8). The curve C has equation $y = \dfrac{15x}{|x|+4}$ and the line l has equation $y = x - 2$.

(a) Solve $x - 2 > \dfrac{15x}{|x|+4}$. The denominator is always positive, so multiply straight through. For $x \geq 0$: $(x-2)(x+4) > 15x$ gives $x^2 - 13x - 8 > 0$, so $x > \frac{13+\sqrt{201}}{2}$ (the other root is negative). For $x < 0$: $(x-2)(4-x) > 15x$ gives $-x^2 - 9x - 8 > 0$, i.e. $(x+1)(x+8) < 0$, so $-8 < x < -1$. Together: $-8 < x < -1$ or $x > \frac{13+\sqrt{201}}{2}$.

(b) Solve $|x-2| > \left|\dfrac{15x}{|x|+4}\right|$. Both sides are non-negative, so square and multiply by $(|x|+4)^2$: the inequality becomes $PQ > 0$, with $P = (x-2)(|x|+4) - 15x$ (the expression from part (a)) and $Q = (x-2)(|x|+4) + 15x$. So P and Q must have the **same sign**. It is not enough to have "$P > 0$ or $Q < 0$". For $x \geq 0$: $P = x^2 - 13x - 8$ and $Q = x^2 + 17x - 8$. Both are positive for $x > \frac{13+\sqrt{201}}{2}$, and both are negative for $0 \leq x < \frac{\sqrt{321}-17}{2}$. For $x < 0$: $Q = -x^2 + 21x - 8$ is negative for every $x < 0$, so we need $P < 0$, which is $(x+1)(x+8) > 0$: $x < -8$ or $-1 < x < 0$. Together: $x < -8$ or $-1 < x < \frac{\sqrt{321}-17}{2}$ or $x > \frac{13+\sqrt{201}}{2}$. (Check $x = -5$: $|x-2| = 7$ but $\left|\frac{-75}{9}\right| = 8.3$, so -5 is correctly excluded.) The "hence" is real: part (a)'s factor P does half the work.

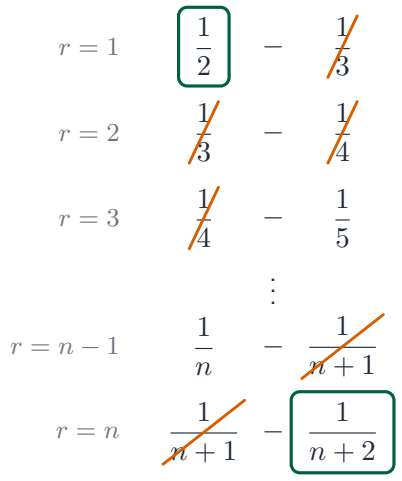


Top: the curve $y = 15x/(|x| + 4)$ with its horizontal asymptotes at ± 15, the line $y = x - 2$, and the intervals where the line is above the curve shaded. Below: both sides' moduli – the reflected curve $|15x/(|x| + 4)|$ and the V of $|x - 2|$, which cross at $x = -8$, -1, $(\sqrt{321} - 17)/2$ and $(13 + \sqrt{201})/2$, with the three solution intervals shaded.

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The method of differences

The **method of differences** 差分法 sums a series by splitting the summand (usually via **partial fractions** 部分分式) so that each term cancels part of a later one – the middle collapses and only the first and last fragments survive. Write out $r = 1$, $r = 2$, $r = n - 1$, $r = n$ explicitly and watch what cancels; that display earns the marks.



each struck $-$ term cancels the $+$ term on the next line; only the boxed first and last survive:

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$$

The method of differences: write out the first and last rows and watch the middle cancel

Worked example – telescoping to a closed form (WFM02 June 2025 Q5). Show $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$. Split:

$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}.$$

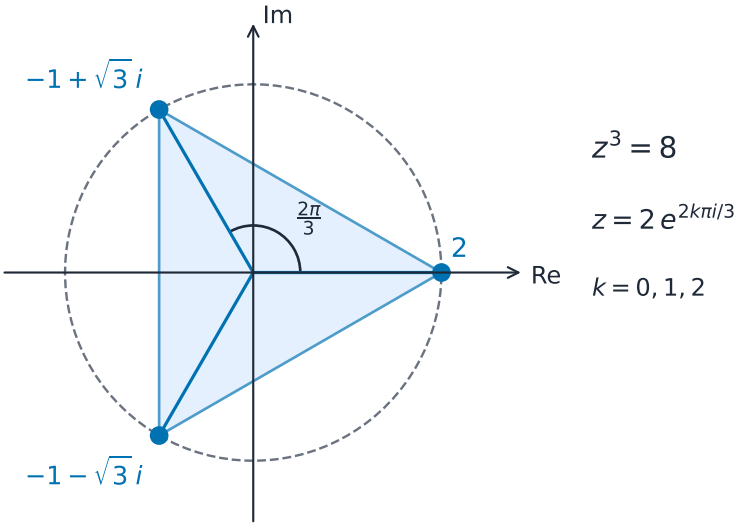
The sum is $\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$: everything from $\frac{1}{3}$ to $\frac{1}{n+1}$ cancels, leaving $\frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$. (Check $n = 1$: both sides $\frac{1}{6}$.) This is also how inequality follow-ups are set: $\dfrac{n}{2(n+2)} > 0.49$ needs $n > 98$, so the smallest is $n = 99$ – the algebra, not a table of values.

Euler's relation and de Moivre's theorem

Euler's relation 欧拉关系式 $e^{i\theta} = \cos\theta + i\sin\theta$ turns the polar form into an exponential: $z = re^{i\theta}$, and $z^n = r^n e^{in\theta}$ becomes pure arithmetic. Read off the companions $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$. **De Moivre's theorem** 棣莫弗定理 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ follows, and is provable for every positive integer n by induction: true for $n = 1$; if true for k, multiplying $(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

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and expanding with the addition formulae gives $\cos(k+1)\theta + i\sin(k+1)\theta$. For negative n, take reciprocals and note $\cos(-n\theta) + i\sin(-n\theta)$ has modulus 1 and the right argument.



The nth roots are equally spaced round a circle

Worked example – a multiple-angle identity (WFM02 June 2025 Q9). Expand the imaginary part of $(\cos\theta + i\sin\theta)^5$ by the binomial theorem:

$$\text{Im} = 5c^4 s - 10c^2 s^3 + s^5 \quad (c = \cos\theta,\ s = \sin\theta).$$

De Moivre says this equals $\sin 5\theta$. Substituting $c^2 = 1 - s^2$: $5s(1-s^2)^2 - 10s^3(1-s^2) + s^5 = 16s^5 - 20s^3 + 5s$, so

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Setting $5\theta = \pi$ gives $16s^4 - 20s^2 + 5 = 0$, so $s^2 = \frac{5\pm\sqrt{5}}{8}$ and $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$ – take the smaller root because $\sin\frac{\pi}{5} < \sin\frac{3\pi}{5}$, and indeed $\sin\frac{3\pi}{5}$ is the larger root.

Worked example – fifth roots (WFM02 June 2025 Q1). Solve $z^5 = -2\sqrt{2} - 2\sqrt{6}i$. Modulus $\sqrt{8+24} = \sqrt{32} = 4\sqrt{2}$; the argument is $-\frac{2\pi}{3}$ (third quadrant, $\tan = \sqrt{3}$). So

$$z = (4\sqrt{2})^{1/5}\, e^{i(-2\pi/15 + 2\pi k/5)} = \sqrt{2}\, e^{i\left(-\frac{2\pi}{15} + \frac{2\pi k}{5}\right)}, \quad k = 0, 1, 2, 3, 4,$$

five roots spaced $\frac{2\pi}{5}$ apart on the circle $|z| = \sqrt{2}$. Use the nth-root pattern $\sqrt[n]{r}\, e^{i(\theta+2\pi k)/n}$ – never decimal approximations.

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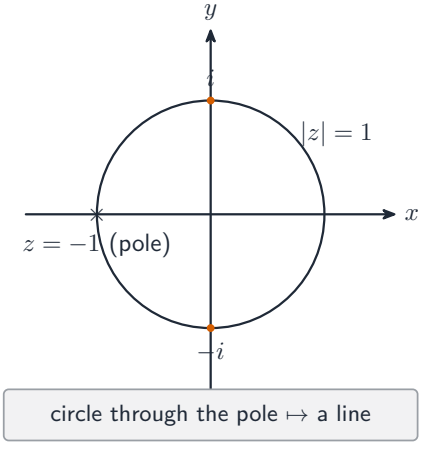
Loci, regions and z→w transformations

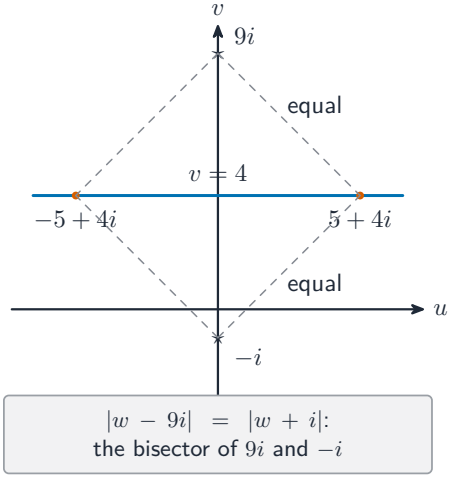
Locus 轨迹 toolkit in the Argand diagram: $|z-a|=b$ is a circle (centre a, radius b); $|z-a|=|z-b|$ is the **perpendicular bisector** 垂直平分线 of a and b; $|z-a|=k|z-b|$ with $k\neq 1$ is a circle (square both sides and collect terms); $\arg(z-a)=\beta$ is a half-line from a. Regions replace $=$ by $<$ or $>$: test one convenient point (the origin rarely lies on the locus). For **Möbius transformations** 莫比乌斯变换 $w=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$, first choose the useful algebraic form rather than relying only on substitution point-by-point: make z the subject first, then take moduli (or arguments) of both sides. When $c\neq 0$, a circle through the pole $z=-\frac{d}{c}$ maps to a straight line; a circle not through the pole maps to a circle.

> **Worked example – image of a circle (WFM02 June 2025 Q3).** $w=\dfrac{9i-iz}{1+z}$ $(z\neq -1)$ maps P, where $|z|=1$, to a line l. Making z the subject: $z(w+i)=9i-w$, so $z=\dfrac{9i-w}{w+i}$. Then $|z|=1$ forces

$$|9i-w|=|w+i| \implies u^2+(v-9)^2=u^2+(v+1)^2 \implies 81-18v=2v+1,$$

so $20v=80$ and l is $v=4$ – the perpendicular bisector of $9i$ and $-i$, exactly as the equal-moduli reading predicts. Spot-check with the endpoints of the vertical diameter: $z=i\mapsto 5+4i$ and $z=-i\mapsto -5+4i$; both land on $v=4$.





Left: the z-plane with the unit circle $|z|=1$ and the pole $z=-1$ marked. Right: the w-plane with the image line $v=4$ and the images of $z=i$ and $z=-i$ lying on it.

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First order differential equations

Separable equations are assumed. The new standard form is $\dfrac{dy}{dx} + Py = Q$ with P, Q functions of x: multiply by the **integrating factor** 积分因子 $e^{\int P\,dx}$, chosen so the left side becomes $\dfrac{d}{dx}\left(ye^{\int P\,dx}\right)$, then integrate once. For a **given substitution** (say $z = f(y)$), compute $\dfrac{dz}{dx}$ by the chain rule, rewrite the equation in z and x, solve the (now linear or separable) equation, and substitute back – the question names the target form, so end in it.



A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Worked example – integrating factor warm-up. $\dfrac{dy}{dx} + 2y = e^{-x}$: $e^{\int 2\,dx} = e^{2x}$, so $\dfrac{d}{dx}(ye^{2x}) = e^{x}$ and $ye^{2x} = e^{x} + C$, i.e. $y = e^{-x} + Ce^{-2x}$.

Worked example – a substitution then an integrating factor (WFM02 June 2025 Q6). Equation (I) is $2\dfrac{dy}{dx} + y\cot x + y^{3}\tan x\sec x = 0$ on $0 < x < \frac{\pi}{2}$.

(a) With $z = \frac{1}{y^{2}}$, $\dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$, so $2\dfrac{dy}{dx} = -y^{3}\dfrac{dz}{dx}$. Substituting and dividing by y^{3}: $-\dfrac{dz}{dx} + y^{-2}\cot x + \tan x\sec x = 0$, i.e.

$$\frac{dz}{dx} - z\cot x = \tan x\sec x \quad \text{(II)}.$$

(b) The integrating factor is $e^{-\int \cot x\,dx} = \csc x$, so $\dfrac{d}{dx}(z\csc x) = \csc x\tan x\sec x =$

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$\frac{1}{\cos^2 x} = \sec^2 x$. Hence $z \csc x = \tan x + C$, giving $z = \sin x \tan x + C \sin x$ and

$$y^2 = \frac{1}{\sin x \tan x + C \sin x}.$$

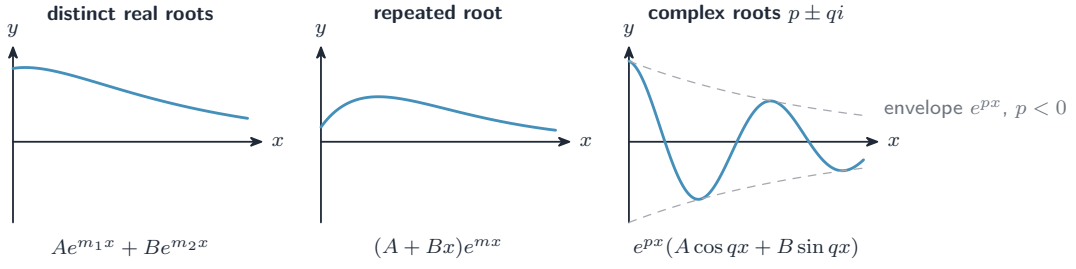
(c) At $x = \frac{\pi}{6}$: $y^2 = \frac{4\sqrt{3}}{3}$ so $z = \frac{\sqrt{3}}{4} = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} + \frac{C}{2}$, forcing $C = \frac{1}{2\sqrt{3}}$. At $x = \frac{\pi}{3}$: $z = \frac{\sqrt{3}}{2} \cdot \sqrt{3} + \frac{1}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$, so $y^2 = \frac{4}{7}$ and $y = \pm\frac{2}{\sqrt{7}}$. Both signs are answers – y^2, not y, was determined.

Second order differential equations

For $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, the general solution is CF + PI: the **complementary function** 余函数 solves the homogeneous equation via the **auxiliary equation** 辅助方程 $am^2 + bm + c = 0$; the **particular integral** 特积分 is any single solution of the full equation.



A pendulum: small swings obey a second-order differential equation, simple harmonic motion



The roots of the auxiliary equation set the shape of the complementary function; here every root has a negative real part.

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Auxiliary equation	Complementary function
distinct real roots $m_1 \neq m_2$	$y = Ae^{m_1 x} + Be^{m_2 x}$
repeated root m	$y = (A + Bx)e^{mx}$
complex roots $p \pm qi$	$y = e^{px}(A\cos qx + B\sin qx)$

Right-hand side $f(x)$	Trial PI
ke^{px}	λe^{px}; if p is a root, use $\lambda x e^{px}$ (again if repeated)
$a + bx(+cx^2)$	$\lambda + \mu x(+\nu x^2)$ – every degree present
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

Given initial or boundary conditions, find the constants last, from the full general solution.

> **Worked example – resonance, then conditions (WFM02 June 2025 Q4).** Given that $y = \lambda x e^{3x}$ is a particular integral of $4\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 3y = 78e^{3x}$:

(a) $y' = \lambda e^{3x}(1 + 3x)$ and $y'' = \lambda e^{3x}(6 + 9x)$, so the left side is $\lambda e^{3x}\big(24 + 36x - 11 - 33x - 3x\big) = 13\lambda e^{3x}$. Matching $78e^{3x}$ gives $\lambda = 6$. (The plain trial λe^{3x} would give $0 \cdot e^{3x}$ – hence the extra x.)

(b) Auxiliary equation $4m^2 - 11m - 3 = (4m + 1)(m - 3) = 0$, so $m = -\frac{1}{4}$ or 3: $y = Ae^{-x/4} + Be^{3x} + 6xe^{3x}$.

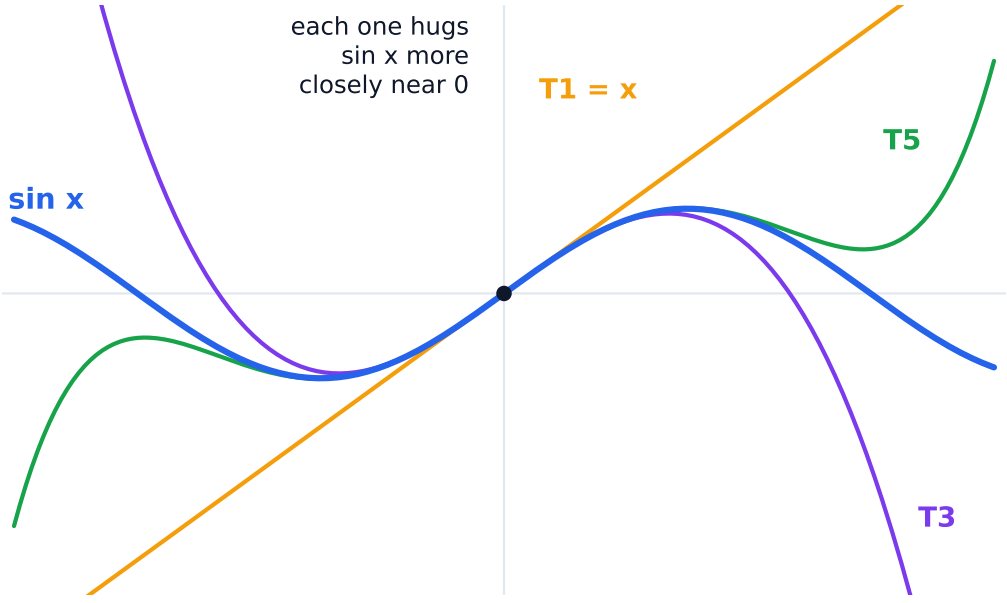
(c) With $y = \frac{9}{2}$ and $y' = 0$ at $x = 0$: $A + B = \frac{9}{2}$ and $-\frac{A}{4} + 3B + 6 = 0$. Solving, $A = 6$, $B = -\frac{3}{2}$, so

$$y = 6e^{-x/4} - \tfrac{3}{2}e^{3x} + 6xe^{3x}.$$

Maclaurin and Taylor series

The **Maclaurin series** 麦克劳林级数 of f is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ – differentiate as many times as the question asks (third and higher derivatives are examinable in their own right). The **Taylor series** 泰勒级数 about $x = a$ replaces x by the offset: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$. For a differential equation, differentiate the equation itself to climb the derivative ladder: $y(a)$ and $y'(a)$ feed the equation for $y''(a)$, then the derivative of the equation for $y'''(a)$, and so on – never re-derive, differentiate.

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The curves compare $\sin x$ *with* $T_1 = x$, $T_3 = x - x^3/6$ *and* $T_5 = x - x^3/6 + x^5/120$.

Worked example – a series solution about a non-zero centre (WFM02 June 2025 Q2). Given $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = e^{2x/9}$ with $y = 2$ and $y' = -1$ at $x = 3$:

at $x = 3$: $y'' = 4y' + 5y + e^{2/3} = -4 + 10 + e^{2/3} = 6 + e^{2/3}$, so the quadratic term is $\frac{1}{2}(6 + e^{2/3})(x-3)^2 = (3 + \frac{1}{2}e^{2/3})(x-3)^2$. Differentiating the equation, $y''' = 4y'' + 5y' + \frac{2}{9}e^{2x/9}$, so at $x = 3$: $y''' = 4(6 + e^{2/3}) - 5 + \frac{2}{9}e^{2/3} = 19 + \frac{38}{9}e^{2/3}$, and the cubic term is $\frac{1}{6}y'''(x-3)^3$. The series is

$$y = 2 - (x-3) + \left(3 + \tfrac{1}{2}e^{2/3}\right)(x-3)^2 + \left(\tfrac{19}{6} + \tfrac{19}{27}e^{2/3}\right)(x-3)^3.$$

Polar coordinates and areas

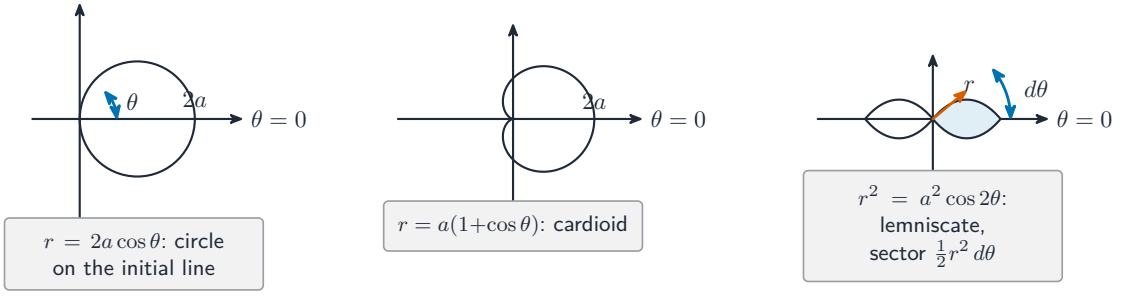
In **polar coordinates** 极坐标 a point is (r, θ) with $r \geq 0$ measured from the **initial line** 初始线 (the positive x-axis, θ anticlockwise). Convert with $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$: for example $r = 4\cos\theta$ gives $r^2 = 4r\cos\theta$, i.e. $x^2 + y^2 = 4x$ – the circle $(x-2)^2 + y^2 = 4$. Know the sketch staples: $r = 2a\cos\theta$ (circle on the initial line through the pole), $r = k\theta$ (spiral), $r = a(1 + \cos\theta)$ (**cardioid** 心形线), $r = a\cos 2\theta$ (four petals), $r^2 = a^2\cos 2\theta$ (**lemniscate** 双纽线, existing only where $\cos 2\theta \geq 0$). The area swept from $\theta = \alpha$ to β is $\frac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$; tangents parallel to the initial line occur where $\dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta = 0$.

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A spiral staircase: spirals are described most simply in polar coordinates

Worked example — exact areas. The whole **lemniscate** $r^2 = 8\cos 2\theta$: by symmetry the right loop is $\frac{1}{2}\int_{-\pi/4}^{\pi/4} 8\cos 2\theta\, d\theta = 4\left[\frac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = 4$, so both loops enclose total area 8. The **cardioid** $r = 2(1+\cos\theta)$: $\frac{1}{2}\int_{-\pi}^{\pi} 4(1+\cos\theta)^2 d\theta = 2\int_{-\pi}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta = 2(2\pi+0+\pi) = 6\pi$, since $\int_{-\pi}^{\pi}\cos^2\theta\, d\theta = \pi$.



Three polar staples drawn on polar grids: the circle $r = 2a\cos\theta$ through the pole, the cardioid $r = a(1+\cos\theta)$, and the lemniscate $r^2 = a^2\cos 2\theta$ with one loop shaded and the sector area element $\frac{1}{2}r^2\,d\theta$ marked.

Checklist before you call this unit done

- Solve modulus inequalities by case split and by squaring; handle $|x|$ inside denominators; report dead branches; reuse an earlier part when "hence" appears.
- Sum finite series by partial fractions and cancellation, writing the $r = 1, 2, n-1, n$ lines; close the follow-up inequality by algebra.

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- Use $e^{i\theta}$, the exponential forms of cos and sin, and de Moivre (proved for integer n) to expand multiple angles and powers, and to find all nth roots in exact polar form.
- Sketch circles, bisectors, Apollonius circles and argument half-lines; test regions; make z the subject of a Möbius transformation and equate moduli or arguments.
- Solve $\frac{dy}{dx} + Py = Q$ with an integrating factor; execute given substitutions and answer in the form demanded, minding $\pm$ from y^2.
- Solve second order equations via auxiliary equation and trial PI (including the multiply-by-x resonance), then fit initial conditions.
- Build Maclaurin and Taylor series, including series solutions of differential equations by differentiating the equation.
- Sketch polar staples, convert to and from Cartesian, and evaluate exact areas with $\frac{1}{2}\int r^2\,d\theta$.

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Further Pure Mathematics 3

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 3: hyperbolic functions, conics, vectors and matrices

Further Pure Mathematics 3 (FP3, paper WFM03/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the last pure unit: P1–P4 and FP1 are assumed and may be tested. It adds the hyperbolic functions and their inverses, the ellipse and hyperbola with their focus-directrix properties, the remaining differentiation and integration techniques (reduction formulae, arc length, surfaces of revolution), vector products with lines and planes in three dimensions, and matrix algebra up to eigenvalues and diagonalisation. This reference covers all of it, with the June 2025 paper as the worked spine.

Must-know for FP3 (assumed, not re-taught):

Family	Formulae
calculus	product, quotient and chain rules; integration by parts and by substitution (P4, FP2)
trigonometry	$\sec^2\theta \equiv 1+\tan^2\theta$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, double angles
vectors (P4)	$\mathbf{a}\cdot\mathbf{b} = a_1b_1+a_2b_2+a_3b_3$, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$
matrices (FP1)	2×2 inverse, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, transformation matrices
exponentials	$e^{\ln x} = x$, quadratic-in-e^x substitutions, ln laws

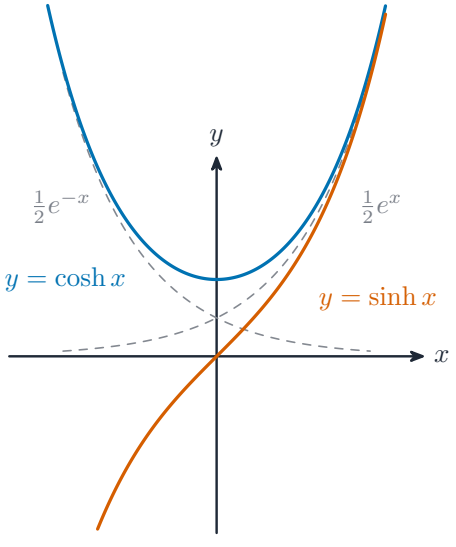
Hyperbolic functions

The **hyperbolic functions** 双曲函数 are built from exponentials: $\cosh x = \frac{1}{2}(e^x+e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, with $\tanh x = \dfrac{\sinh x}{\cosh x}$. The graph of cosh is the **catenary** 悬链线, even with minimum $(0,1)$; sinh is odd through the origin; tanh is odd and trapped between the **asymptote** 渐近线 lines $y=\pm 1$. The two identities to derive on demand are $\cosh^2 x - \sinh^2 x \equiv 1$ and $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$; the sum-to-product identities mirror the trigonometric ones.

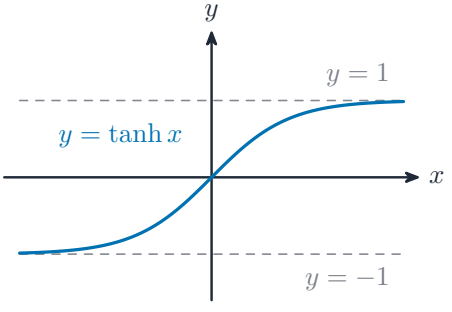
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A chain hanging between two posts sags into a catenary, the curve $y = a\cosh(x/a)$



$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$



$$\tanh x = \frac{\sinh x}{\cosh x}, \text{ bounded by } \pm 1$$

Left: $y = \cosh x$ built from the dashed half-exponentials, with $y = \sinh x$. Right: $y = \tanh x$ between its asymptotes $y = \pm 1$.

Worked example – an identity from the definitions (WFM03 June 2025 Q1). Show that $\cosh 6x + \cosh 4x = 2\cosh 5x \cosh x$. From the left:

$$\tfrac{1}{2}(e^{6x} + e^{-6x}) + \tfrac{1}{2}(e^{4x} + e^{-4x}) = \tfrac{1}{2}(e^{5x} + e^{-5x})(e^{x} + e^{-x}) = 2\cosh 5x \cosh x,$$

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pairing $e^{6x} = e^{5x}e^{x}$ with $e^{-6x} = e^{-5x}e^{-x}$ and likewise for the $4x$ terms – every step is bookkeeping with products $e^{\pm x}e^{\pm 5x}$, which is why the marks demand the exponential forms before any cancellation. Then solve $\cosh 6x + \cosh 4x = 8\cosh x$: the identity collapses it to $\cosh x(2\cosh 5x - 8) = 0$, and cosh is never zero, so $\cosh 5x = 4$, giving $5x = \pm\operatorname{arcosh} 4 = \pm\ln(4+\sqrt{15})$ and $x = \pm\frac{1}{5}\ln(4+\sqrt{15})$.

> **Worked example – a cosh–sinh equation (original).** Solve $2\cosh x + \sinh x = 2$: the exponential definitions give $2\cdot\frac{1}{2}(e^{x}+e^{-x}) + \frac{1}{2}(e^{x}-e^{-x}) = 2$, so $3e^{x} + e^{-x} = 4$, i.e. $3e^{2x} - 4e^{x} + 1 = 0$, i.e. $(3e^{x}-1)(e^{x}-1) = 0$, and $x = 0$ or $x = -\ln 3$. Check both in the original equation – no squaring is ever needed.

Inverse hyperbolic functions

The inverses exist as single functions because sinh and tanh are one-to-one and cosh is one-to-one for $x \geq 0$; their graphs are reflections in $y = x$ of the restricted originals. The **logarithmic equivalents** must be derivable, not just quoted:

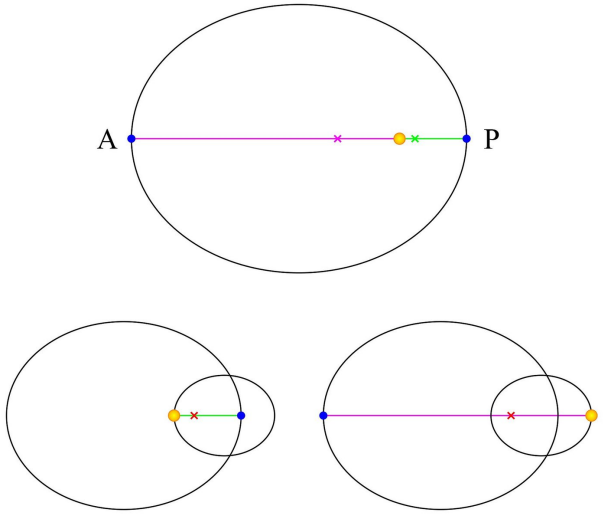
$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right), \quad \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)\ (x \geq 1), \quad \operatorname{artanh} x = \tfrac{1}{2}\ln\tfrac{1+x}{1-x}\ (|x| < 1).$$

Derivation pattern (arsinh). Set $w = \sinh x = \frac{1}{2}(e^{x}-e^{-x})$: then $e^{2x} - 2we^{x} - 1 = 0$, so $e^{x} = w + \sqrt{w^2+1}$ (the negative root is rejected because $e^{x} > 0$) and $x = \ln(w + \sqrt{w^2+1})$. The same quadratic-in-$e^{x}$ structure solves every mixed hyperbolic equation exactly.

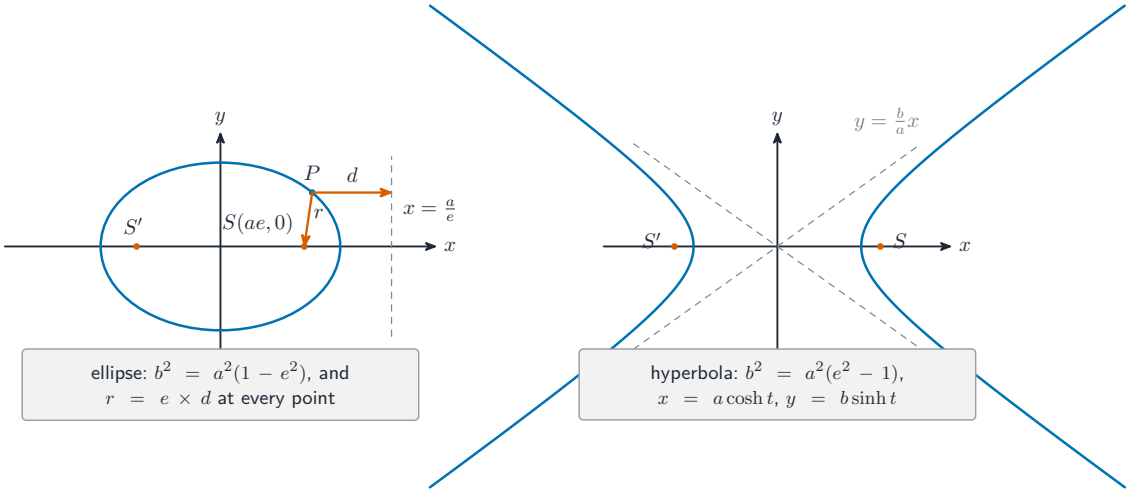
The ellipse and the hyperbola

The **ellipse** 椭圆 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has parametric form $x = a\cos t$, $y = b\sin t$; the **hyperbola** 双曲线 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has $x = a\sec t$, $y = b\tan t$ and also $x = a\cosh t$, $y = b\sinh t$ (the right branch). The **eccentricity** 离心率 e ties everything to the **focus** 焦点 –**directrix** 准线 property $PS = e\cdot PN$ (distance to focus = $e\times$ distance to directrix): for the non-circular ellipse with $a > b > 0$, $b^2 = a^2(1-e^2)$, foci $(\pm ae, 0)$, directrices $x = \pm\dfrac{a}{e}$; for the hyperbola $b^2 = a^2(e^2-1)$ with the same focus and directrix formulas.

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Planets and satellites move in ellipses, with the attracting body at one focus

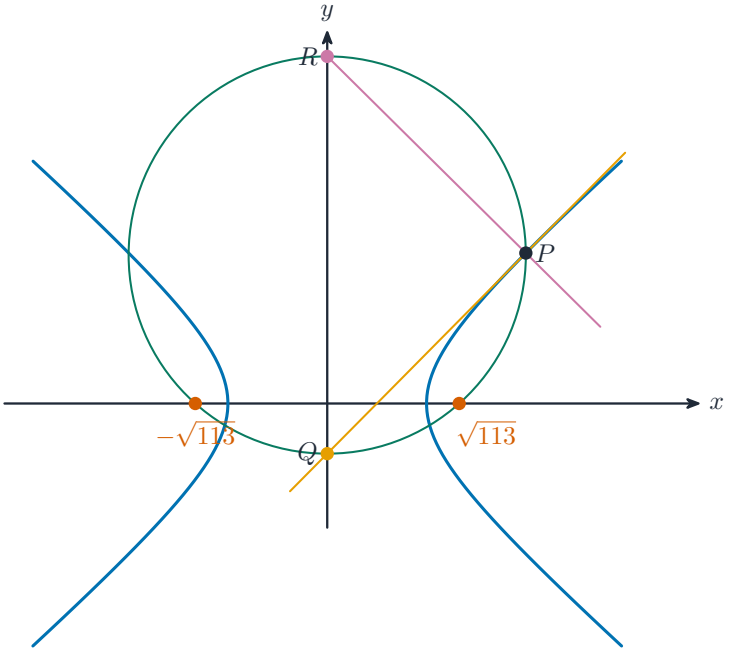


Left: the ellipse with foci $(\pm ae,\,0)$, the directrix $x = a/e$, and the ratio property $PS = e \cdot PN$ marked at a point P. Right: the hyperbola with both branches, asymptotes $y = \pm(b/a)x$ and foci.

Worked example – eccentricity and foci (WFM03 June 2025 Q9a). For H: $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1$: $a^2 = 64$, $b^2 = 49$, so $49 = 64(e^2 - 1)$, $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$; the foci are $(\pm\sqrt{113}, 0)$.

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Substituting $y = mx+c$ into the conic and forcing a zero discriminant gives the tangent condition: $c^2 = a^2m^2+b^2$ for the ellipse, $c^2 = a^2m^2-b^2$ for the hyperbola (so $|m| > \frac{b}{a}$ is needed there). At a parametric point, differentiate instead – no discriminant required. Loci problems replay FP1: distances to foci, midpoints, or a fixed ratio.



tangent meets the y-axis at Q;
normal meets it at R;
$qr = -113$ for every θ, so the circle on QR passes through both foci $(\pm\sqrt{113}, 0)$

drawn for $\theta = \pi/3$

The theta cancels: both foci lie on the circle

Worked example – tangent, normal and a focal circle (WFM03 June 2025 Q9b–e). At $P(8\sec\theta, 7\tan\theta)$ on H: $\dfrac{dy}{dx} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta} = \dfrac{7\sec\theta}{8\tan\theta}$, so the tangent is

$$\tfrac{x}{8}\sec\theta - \tfrac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta = 1,$$

meeting the y-axis at $Q = (0, -7\cot\theta)$; the given normal $8x\cos\theta + 7y\cot\theta = 113$ meets it at $R = (0, \frac{113}{7}\tan\theta)$. The circle on diameter QR (endpoints q, r on the y-axis) has equation $x^2 + (y - \frac{q+r}{2})^2 = (\frac{r-q}{2})^2$, which contains a point $(x, 0)$ exactly when $x^2 = -qr$. Here $qr = -7\cot\theta \cdot \frac{113}{7}\tan\theta = -113$ for every θ, and $113 = (\sqrt{113})^2$ is exactly the squared focal distance – so both foci $(\pm\sqrt{113}, 0)$ lie on the circle. The θ cancelling is the point of the question.

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Differentiating hyperbolic and inverse functions

$\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$, and from the inverses: $\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{x^2+1}}$, $\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2}$ – each provable by inverting $y = f(x)$ and differentiating implicitly, and each valid only on its domain.

> **Worked example – inverse derivatives on a restricted domain (WFM03 June 2025 Q4).** For $y = \operatorname{arsinh} x + \operatorname{arsinh}\frac{1}{x}$ with $x > 0$:
>
> $$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}} = \frac{x-1}{x\sqrt{1+x^2}},$$
>
> where $x > 0$ turns $\sqrt{x^2}$ into x – the domain condition is doing real work. Setting $\frac{dy}{dx} = 0$ gives $x = 1$ and $y = 2\operatorname{arsinh} 1 = 2\ln(1+\sqrt{2})$.

Integrating hyperbolic and inverse functions

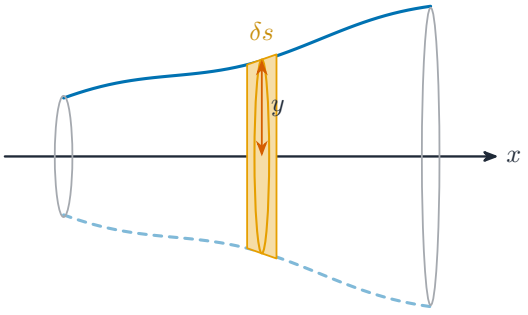
The standard results mirror the derivatives. Complete the square first for surd denominators, and handle inverse functions by parts – take u as the inverse function and 1 as the rest:

> **Worked example – both techniques (WFM03 June 2025 Q2).** (i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$: since $4x^2+8x+9 = 4\left((x+1)^2+\frac{5}{4}\right)$, the integral is $\frac{1}{2}\displaystyle\int \frac{dx}{\sqrt{(x+1)^2+5/4}} = \frac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$ (differentiate back to check). (ii) $\displaystyle\int \operatorname{arcosh} 3x\,dx = x\operatorname{arcosh} 3x - \int \frac{3x}{\sqrt{9x^2-1}}dx = x\operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2-1} + c$.

Substitutions, reduction formulae and surfaces of revolution

Take $a > 0$ below; the displayed inverse-cosh antiderivative uses the branch $x > a$. Four integrals to know cold, each via $x = a\sinh t$ (or $x = a\tan t$, $a\coth$-type splits): $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$. A **reduction formula** 递推公式 expresses I_n through I_{n-1} (or I_{n-2}): integrate by parts with the surd as v, then rewrite the leftover surd integral using the definition of I_n. **Arc length** 弧长 and the **surface of revolution** 旋转曲面 need $s = \displaystyle\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ and $S = 2\pi\displaystyle\int |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (parametrically, $s = \int \sqrt{\dot{x}^2+\dot{y}^2}\,dt$ with $y\sqrt{\dot{x}^2+\dot{y}^2}$ for S).

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band area $\approx 2\pi y\,\delta s$

$$S = 2\pi \int |y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

A surface built from thin bands

Worked example – building a reduction formula (WFM03 June 2025 Q5). With $I_n = \displaystyle\int_1^6 \frac{x^n}{\sqrt{3x-2}}dx$, take $u = x^n$ and $v = \frac{2}{3}\sqrt{3x-2}$:

$$I_n = \left[\tfrac{2}{3}x^n\sqrt{3x-2}\right]_1^6 - \tfrac{2n}{3}\int_1^6 x^{n-1}\sqrt{3x-2}\,dx = \tfrac{8\cdot 6^n - 2}{3} - \tfrac{2n}{3}\left(3I_n - 2I_{n-1}\right),$$

using $\sqrt{3x-2} = \dfrac{3x-2}{\sqrt{3x-2}}$ to expose I_n and I_{n-1}. Collecting: $(3+6n)I_n = 4nI_{n-1} + 8\cdot 6^n - 2$. Seed with $I_0 = \frac{2}{3}(4-1) = 2$ and $I_1 = 6$ (by direct substitution), then $15I_2 = 8(6) + 8(36) - 2 = 334$, and $21I_3 = 12\cdot\frac{334}{15} + 8(216) - 2$, giving $I_3 = \frac{3322}{35}$.

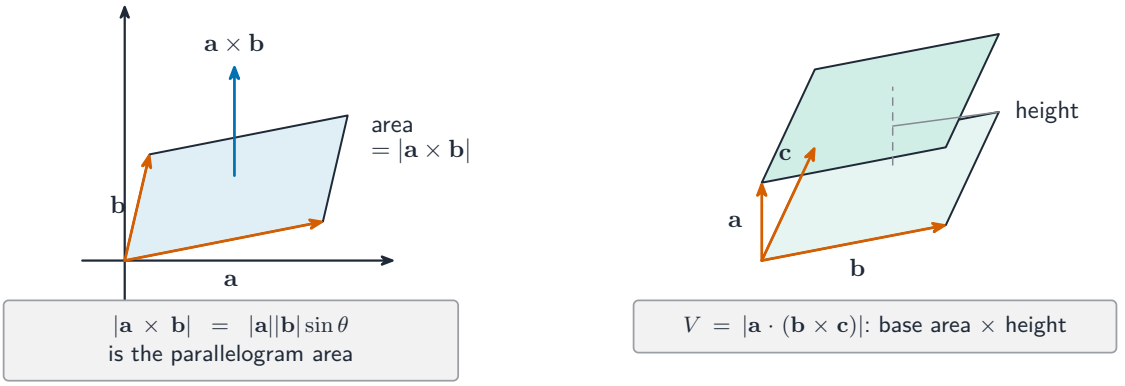
Worked example – a surface with a sinh substitution (WFM03 June 2025 Q7). Rotating $y = \cos 2x$, $0 \le x \le \frac{\pi}{4}$, through 2π about the x-axis (calculator-banned): $S = 2\pi\displaystyle\int_0^{\pi/4} \cos 2x\sqrt{1 + 4\sin^2 2x}\,dx$. With $2\sin 2x = \sinh\theta$, so $4\cos 2x\,dx = \cosh\theta\,d\theta$, the integral becomes $\frac{\pi}{2}\displaystyle\int \cosh^2\theta\,d\theta = \frac{\pi}{4}\Big[\sinh\theta\cosh\theta + \theta\Big]$ between 0 and $\operatorname{arsinh} 2 = \ln(2+\sqrt{5})$, so $S = \frac{\pi}{4}\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$ – the $\cos 2x$ factors cancel exactly, which is why the substitution is supplied.

Vectors: products, lines and planes

The **vector product** 向量积 $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is perpendicular to both factors with $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ – the parallelogram area. The **scalar triple product** 混合积 $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is the signed parallelepiped volume; zero means the three vectors are coplanar. Lines: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ or $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = \mathbf{0}$; planes: $\mathbf{r}\cdot\mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ with $\mathbf{n} = \mathbf{b}\times\mathbf{c}$. Distance from a point to a plane: $\dfrac{|\mathbf{x}\cdot\mathbf{n} - p|}{|\mathbf{n}|}$; shortest distance between **skew lines** 异面直线: project the offset between anchor points onto the common

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normal $\mathbf{b}_1 \times \mathbf{b}_2$.



Left: the vector product as the parallelogram area with its normal. Right: the scalar triple product as the volume of the parallelepiped with the height marked.

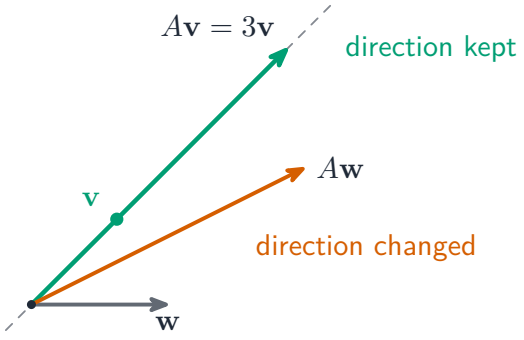
Worked example – planes to lines to distance (WFM03 June 2025 Q8). For $\Pi_1 : \mathbf{r} = (4,2,-1) + \lambda(7,5,-6) + \mu(-3,1,2)$: the normal is $(7,5,-6) \times (-3,1,2) = (16,4,22) = 2(8,2,11)$, and substituting the anchor point gives $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = 32 + 4 - 11 = 25$. Intersecting with $\Pi_2 : x - y + z = 7$: the line direction is $(8,2,11) \times (1,-1,1) = (13,3,-10)$ and a point solving both planes is $(\frac{13}{5}, -\frac{17}{5}, 1)$, so $(\mathbf{r} - (\frac{13}{5}, -\frac{17}{5}, 1)) \times (13,3,-10) = \mathbf{0}$. The line through $A(2,1,3)$ and $B(3,0,2)$ has direction $(1,-1,-1)$; the common normal is $(13,3,-10) \times (1,-1,-1) = (-13,3,-16)$, and the shortest distance is

$$\frac{\left|(-\frac{3}{5}, \frac{22}{5}, 2) \cdot (-13,3,-16)\right|}{\sqrt{434}} = \frac{\left|\frac{39}{5} + \frac{66}{5} - 32\right|}{\sqrt{434}} = \frac{11}{\sqrt{434}}.$$

Further matrix algebra

Determinants and inverses extend to 3×3: expand along any row or column (signs $+ - + - + -$); $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}(\text{adjugate})$, the adjugate being the transpose of the cofactor matrix. **Singular** 奇异 matrices ($\det = 0$) have no inverse and collapse dimension. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. An **eigenvalue** 特征值 λ with **eigenvector** 特征向量 $\mathbf{v}$ satisfies $\mathbf{Av} = \lambda\mathbf{v}$ – row by row, each component of $\mathbf{Av}$ equals λ times the corresponding component of $\mathbf{v}$ (not λ itself). Given $\mathbf{v} \neq \mathbf{0}$, use any nonzero component v_i and calculate $\lambda = (\mathbf{Av})_i / v_i$, including $\lambda = 0$; otherwise solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and back-substitute. For a symmetric $\mathbf{A}$, eigenvectors for distinct eigenvalues are orthogonal, and an orthogonal matrix $\mathbf{P}$ of normalised eigenvectors **diagonalises** 对角化 it: $\mathbf{P}^T\mathbf{AP} = \mathbf{D}$.

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An eigenvector keeps its direction under the matrix; a general vector is turned

Worked example – eigenvalues from a given eigenvector (WFM03 June 2025 Q3). $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ has eigenvector $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Row 1: $1-2+4 = 3 = 3(1)$, so $\lambda = 3$. Row 2: $3 + 2a + b$ must equal $\lambda \times 2 = 6$, and row 3: $a + 2 + b$ must equal $\lambda \times 1 = 3$. So $2a + b = 3$ and $a + b = 1$: subtracting, $a = 2$ and $b = -1$. With $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$: the trace is 2 and, expanding along row 1, $\det \mathbf{M} = 1(-2 + 1) + 1(-3 + 2) + 4(3 - 4) = -1 - 1 - 4 = -6$. The remaining eigenvalues then sum to $2 - 3 = -1$ and multiply to $\frac{-6}{3} = -2$: they are 1 and -2. Eigenvectors: for $\lambda = 1$, solving $(\mathbf{M}-\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $-y+4z = 0$ and $3x+y-z = 0$, so $\mathbf{v} = (-1, 4, 1)$; for $\lambda = -2$, $(\mathbf{M}+2\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $3x-y+4z = 0$ and $2x+y+z = 0$, so $\mathbf{v} = (-1, 1, 1)$ (both row-checked). Trace and determinant are the free verification: $3 + 1 - 2 = 2$ ✓ and $3 \cdot 1 \cdot (-2) = -6$ ✓.

Worked example – singular and inverse with a parameter (WFM03 June 2025 Q6). $\mathbf{A} = \begin{pmatrix} 1 & k & 2 \\ 5 & 3 & -2 \\ 6 & -1 & 4 \end{pmatrix}$: $\det \mathbf{A} = 1(12 - 2) - k(20 + 12) + 2(-5 - 18) = -36-32k$, so $\mathbf{A}$ is singular when $k = -\frac{9}{8}$. The adjugate (transposed cofactors) gives

$$\mathbf{A}^{-1} = \frac{1}{-36 - 32k} \begin{pmatrix} 10 & -4k - 2 & -2k - 6 \\ -32 & -8 & 12 \\ -23 & 6k + 1 & 3 - 5k \end{pmatrix}.$$

Sanity-check the (1,1) entry of the product: $(1, k, 2) \cdot (10, -32, -23) = 10-32k-46 = -36 - 32k$, which cancels against the determinant to give 1 ✓.

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Checklist before you call this unit done

- Define $\cosh, \sinh, \tanh$ from exponentials; derive the core identities; solve mixed equations via e^x quadratics with exact ln answers.
- Derive the $\operatorname{arsinh}, \operatorname{arcosh}, \operatorname{artanh}$ log forms; differentiate inverses with domains respected.
- Use ellipse and hyperbola parametrics; convert between $b^2 = a^2(1-e^2)$ and $a^2(e^2-1)$; place foci and directrices; find tangents/normals at parametric points and via the $y = mx + c$ condition $c^2 = a^2m^2 \pm b^2$.
- Differentiate and integrate hyperbolic and inverse functions, including by parts for $\int$ arcosh-type integrands.
- Know the four surd/rational standard integrals; build reduction formulae by parts and iterate; compute arc lengths and surfaces of revolution exactly.
- Compute $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ with their area and volume meanings; move between plane and line forms; find point-to-plane and skew-line distances.
- Evaluate 3×3 determinants, invert via the adjugate, handle the singular case, find eigenvalues/eigenvectors (normalised when asked) and diagonalise symmetric matrices with an orthogonal $\mathbf{P}$.

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