Further Pure Mathematics 3

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 3: hyperbolic functions, conics, vectors and matrices

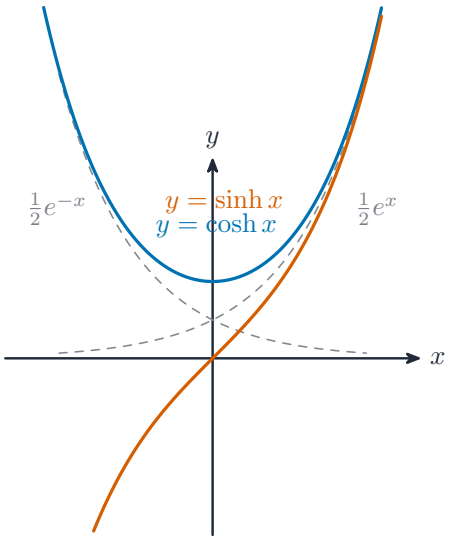
Further Pure Mathematics 3 (FP3, paper WFM03/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the last pure unit: P1–P4 and FP1 are assumed and may be tested. It adds the hyperbolic functions and their inverses, the ellipse and hyperbola with their focus-directrix properties, the remaining differentiation and integration techniques (reduction formulae, arc length, surfaces of revolution), vector products with lines and planes in three dimensions, and matrix algebra up to eigenvalues and diagonalisation. This reference covers all of it, with the June 2025 paper as the worked spine.

Must-know for FP3 (assumed, not re-taught):

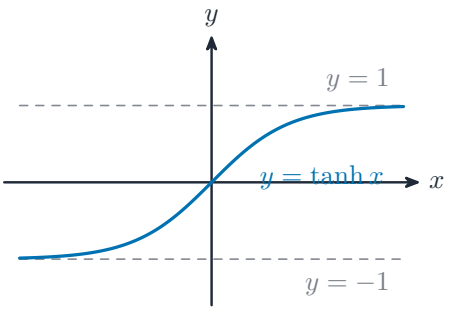
Family	Formulae
calculus	product, quotient and chain rules; integration by parts and by substitution (P4, FP2)
trigonometry	$\sec^2\theta \equiv 1+\tan^2\theta$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, double angles
vectors (P4)	$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$
matrices (FP1)	2×2 inverse, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, transformation matrices
exponentials	$e^{\ln x} = x$, quadratic-in-e^x substitutions, ln laws

Hyperbolic functions

The **hyperbolic functions** are built from exponentials: $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, with $\tanh x = \dfrac{\sinh x}{\cosh x}$. The graph of cosh is the **catenary**, even with minimum $(0,1)$; sinh is odd through the origin; tanh is odd and trapped between the **asymptote** lines $y = \pm 1$. The two identities to derive on demand are $\cosh^2 x - \sinh^2 x \equiv 1$ and $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$; the sum-to-product identities mirror the trigonometric ones.



$$\begin{aligned}\cosh x &= \tfrac{1}{2}(e^x + e^{-x}),\\ \sinh x &= \tfrac{1}{2}(e^x - e^{-x})\end{aligned}$$



$\tanh x = \dfrac{\sinh x}{\cosh x}$, bounded by ± 1

Worked example – an identity from the definitions (WFM03 June 2025 Q1). Show that $\cosh 6x + \cosh 4x = 2\cosh 5x \cosh x$. From the left:

$$\tfrac{1}{2}(e^{6x} + e^{-6x}) + \tfrac{1}{2}(e^{4x} + e^{-4x}) = \tfrac{1}{2}(e^{5x} + e^{-5x})(e^x + e^{-x}) = 2\cosh 5x \cosh x,$$

pairing $e^{6x} = e^{5x}e^x$ with $e^{-6x} = e^{-5x}e^{-x}$ and likewise for the $4x$ terms – every step is bookkeeping with products $e^{\pm x}e^{\pm 5x}$, which is why the marks demand the exponential forms before any cancellation. Then solve $\cosh 6x + \cosh 4x = 8\cosh x$: the identity collapses it to $\cosh x(2\cosh 5x - 8) = 0$, and cosh is never zero, so $\cosh 5x = 4$, giving $5x = \pm \operatorname{arcosh} 4 = \pm \ln(4 + \sqrt{15})$ and $x = \pm\frac{1}{5}\ln(4 + \sqrt{15})$.

Worked example – a cosh–sinh equation (original). Solve $2\cosh x + \sinh x = 2$: the exponential definitions give $2 \cdot \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = 2$, so $3e^x + e^{-x} = 4$, i.e. $3e^{2x} - 4e^x + 1 = 0$, i.e. $(3e^x - 1)(e^x - 1) = 0$, and $x = 0$ or $x = -\ln 3$. Check both in the original equation – no squaring is ever needed.

Inverse hyperbolic functions

The inverses exist as single functions because sinh and tanh are one-to-one and cosh is one-to-one for $x \geq 0$; their graphs are reflections in $y = x$ of the restricted originals. The **logarithmic equivalents** must be derivable, not just quoted:

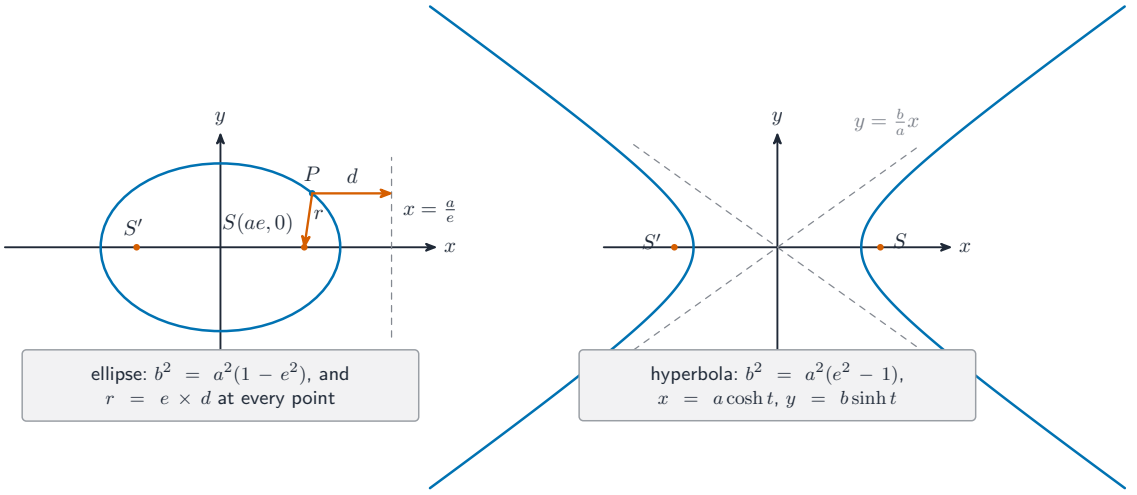
$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right), \quad \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \ (x \geq 1), \quad \operatorname{artanh} x = \tfrac{1}{2}\ln\tfrac{1+x}{1-x} \ (|x| < 1).$$

Derivation pattern (arsinh). Set $w = \sinh x = \frac{1}{2}(e^x - e^{-x})$: then $e^{2x} - 2we^x - 1 = 0$,

so $e^x = w + \sqrt{w^2+1}$ (the negative root is rejected because $e^x > 0$) and $x = \ln(w + \sqrt{w^2+1})$. The same quadratic-in-$e^x$ structure solves every mixed hyperbolic equation exactly.

The ellipse and the hyperbola

The **ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has parametric form $x = a\cos t$, $y = b\sin t$; the **hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has $x = a\sec t$, $y = b\tan t$ and also $x = a\cosh t$, $y = b\sinh t$ (the right branch). The **eccentricity** e ties everything to the **focus–directrix** property $PS = e \cdot PN$ (distance to focus $= e \times$ distance to directrix): for the ellipse $b^2 = a^2(1-e^2)$, foci $(\pm ae, 0)$, directrices $x = \pm\dfrac{a}{e}$; for the hyperbola $b^2 = a^2(e^2-1)$ with the same focus and directrix formulas.



Worked example – eccentricity and foci (WFM03 June 2025 Q9a). For H: $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1$: $a^2 = 64$, $b^2 = 49$, so $49 = 64(e^2-1)$, $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$; the foci are $(\pm\sqrt{113}, 0)$.

Tangents, normals and loci

Substituting $y = mx + c$ into the conic and forcing a zero discriminant gives the tangent condition: $c^2 = a^2m^2 + b^2$ for the ellipse, $c^2 = a^2m^2 - b^2$ for the hyperbola (so $|m| > \frac{b}{a}$ is needed there). At a parametric point, differentiate instead – no discriminant required. Loci problems replay FP1: distances to foci, midpoints, or a fixed ratio.

Worked example – tangent, normal and a focal circle (WFM03 June 2025 Q9b–e). At $P(8\sec\theta, 7\tan\theta)$ on H: $\dfrac{dy}{dx} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta} = \dfrac{7\sec\theta}{8\tan\theta}$, so the tangent is

$$\tfrac{x}{8}\sec\theta - \tfrac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta = 1,$$

meeting the y-axis at $Q = (0, -7\cot\theta)$; the given normal $8x\cos\theta + 7y\cot\theta = 113$ meets it at $R = (0, \frac{113}{7}\tan\theta)$. The circle on diameter QR (endpoints q, r on the y-axis) has equation $x^2 + (y - \frac{q+r}{2})^2 = (\frac{r-q}{2})^2$, which contains a point $(x, 0)$ exactly when $x^2 = -qr$. Here $qr = -7\cot\theta \cdot \frac{113}{7}\tan\theta = -113$ for every θ, and $113 = (\sqrt{113})^2$ is exactly the squared focal distance – so both foci $(\pm\sqrt{113}, 0)$ lie on the circle. The θ cancelling is the point of the question.

Differentiating hyperbolic and inverse functions

$\dfrac{d}{dx}\sinh x = \cosh x$, $\dfrac{d}{dx}\cosh x = \sinh x$, $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$, and from the inverses: $\dfrac{d}{dx}\operatorname{arsinh} x = \dfrac{1}{\sqrt{x^2+1}}$, $\dfrac{d}{dx}\operatorname{arcosh} x = \dfrac{1}{\sqrt{x^2-1}}$, $\dfrac{d}{dx}\operatorname{artanh} x = \dfrac{1}{1-x^2}$ – each provable by inverting $y = f(x)$ and differentiating implicitly, and each valid only on its domain.

Worked example – inverse derivatives on a restricted domain (WFM03 June 2025 Q4). For $y = \operatorname{arsinh} x + \operatorname{arsinh}\dfrac{1}{x}$ with $x > 0$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}} = \frac{x-1}{x\sqrt{1+x^2}},$$

where $x > 0$ turns $\sqrt{x^2}$ into x – the domain condition is doing real work. Setting $\dfrac{dy}{dx} = 0$ gives $x = 1$ and $y = 2\operatorname{arsinh} 1 = 2\ln(1+\sqrt{2})$.

Integrating hyperbolic and inverse functions

The standard results mirror the derivatives. Complete the square first for surd denominators, and handle inverse functions by parts – take u as the inverse function and 1 as the rest:

Worked example – both techniques (WFM03 June 2025 Q2). (i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$: since $4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right)$, the integral is $\displaystyle\frac{1}{2}\int \frac{dx}{\sqrt{(x+1)^2 + 5/4}} = \frac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$ (differentiate back to check). (ii) $\displaystyle\int \operatorname{arcosh} 3x\, dx = x\operatorname{arcosh} 3x - \int \frac{3x}{\sqrt{9x^2-1}}dx = x\operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2-1} + c.$

Substitutions, reduction formulae and surfaces of revolution

Four integrals to know cold, each via $x = a\sinh t$ (or $x = a\tan t$, $a\coth$-type splits): $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2+a^2} = \tfrac{1}{a}\arctan\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2-a^2} = \tfrac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$. A **reduction formula** expresses I_n through I_{n-1} (or I_{n-2}): integrate by parts with the surd as v, then rewrite the leftover surd integral using the definition of I_n. **Arc length** and the **surface of revolution** need $s = \displaystyle\int \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx$ and $S = 2\pi\displaystyle\int y\sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx$ (parametrically, $s = \int \sqrt{\dot{x}^2+\dot{y}^2}\,dt$ with $y\sqrt{\dot{x}^2+\dot{y}^2}$ for S).

Worked example – building a reduction formula (WFM03 June 2025 Q5). With $I_n = \displaystyle\int_1^6 \frac{x^n}{\sqrt{3x-2}}dx$, take $u = x^n$ and $v = \tfrac{2}{3}\sqrt{3x-2}$:

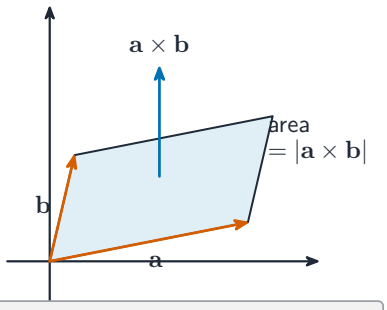
$$I_n = \left[\tfrac{2}{3}x^n\sqrt{3x-2}\right]_1^6 - \tfrac{2n}{3}\int_1^6 x^{n-1}\sqrt{3x-2}\,dx = \tfrac{8\cdot 6^n-2}{3} - \tfrac{2n}{3}\left(3I_n - 2I_{n-1}\right),$$

using $\sqrt{3x-2} = \dfrac{3x-2}{\sqrt{3x-2}}$ to expose I_n and I_{n-1}. Collecting: $(3+6n)I_n = 4nI_{n-1} + 8\cdot 6^n - 2$. Seed with $I_0 = \tfrac{2}{3}(4-1) = 2$ and $I_1 = 6$ (by direct substitution), then $15I_2 = 8(6) + 8(36) - 2 = 334$, and $21I_3 = 12\cdot\tfrac{334}{15} + 8(216) - 2$, giving $I_3 = \tfrac{3322}{35}$.

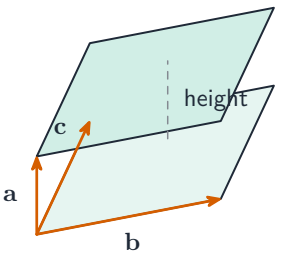
Worked example – a surface with a sinh substitution (WFM03 June 2025 Q7). Rotating $y = \cos 2x$, $0 \le x \le \tfrac{\pi}{4}$, through 2π about the x-axis (calculator-banned): $S = 2\pi\displaystyle\int_0^{\pi/4}\cos 2x\sqrt{1+4\sin^2 2x}\,dx$. With $2\sin 2x = \sinh\theta$, so $4\cos 2x\,dx = \cosh\theta\,d\theta$, the integral becomes $\tfrac{\pi}{2}\displaystyle\int\cosh^2\theta\,d\theta = \tfrac{\pi}{4}\Big[\sinh\theta\cosh\theta + \theta\Big]$ between 0 and $\operatorname{arsinh} 2 = \ln(2+\sqrt{5})$, so $S = \tfrac{\pi}{4}\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$ – the $\cos 2x$ factors cancel exactly, which is why the substitution is supplied.

Vectors: products, lines and planes

The **vector product** $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is perpendicular to both factors with $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ – the parallelogram area. The **scalar triple product** $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is the signed parallelepiped volume; zero means the three vectors are coplanar. Lines: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ or $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = \mathbf{0}$; planes: $\mathbf{r}\cdot\mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ with $\mathbf{n} = \mathbf{b}\times\mathbf{c}$. Distance from a point to a plane: $\dfrac{|\mathbf{x}\cdot\mathbf{n} - p|}{|\mathbf{n}|}$; shortest distance between **skew lines**: project the offset between anchor points onto the common normal $\mathbf{b}_1\times\mathbf{b}_2$.



$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$
is the parallelogram area



$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$: base area × height

Worked example – planes to lines to distance (WFM03 June 2025 Q8). For $\Pi_1 : \mathbf{r} = (4, 2, -1) + \lambda(7, 5, -6) + \mu(-3, 1, 2)$: the normal is $(7, 5, -6) \times (-3, 1, 2) = (16, 4, 22) = 2(8, 2, 11)$, and substituting the anchor point gives $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = 32 + 4 - 11 = 25$. Intersecting with $\Pi_2 : x - y + z = 7$: the line direction is $(8, 2, 11) \times (1, -1, 1) = (13, 3, -10)$ and a point solving both planes is $(\frac{13}{5}, -\frac{17}{5}, 1)$, so $(\mathbf{r} - (\frac{13}{5}, -\frac{17}{5}, 1)) \times (13, 3, -10) = \mathbf{0}$. The line through $A(2, 1, 3)$ and $B(3, 0, 2)$ has direction $(1, -1, -1)$; the common normal is $(13, 3, -10) \times (1, -1, -1) = (-13, 3, -16)$, and the shortest distance is

$$\frac{\left|\left(-\frac{3}{5}, \frac{22}{5}, 2\right) \cdot (-13, 3, -16)\right|}{\sqrt{434}} = \frac{\left|\frac{39}{5} + \frac{66}{5} - 32\right|}{\sqrt{434}} = \frac{11}{\sqrt{434}}.$$

Further matrix algebra

Determinants and inverses extend to 3×3: expand along any row or column (signs $+ - + - + -$); $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}(\text{adjugate})$, the adjugate being the transpose of the cofactor matrix. **Singular** matrices ($\det = 0$) have no inverse and collapse dimension. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. An **eigenvalue** λ with **eigenvector** $\mathbf{v}$ satisfies $\mathbf{Av} = \lambda\mathbf{v}$ – row by row, each component of $\mathbf{Av}$ equals λ times the corresponding component of $\mathbf{v}$ (not λ itself). Given $\mathbf{v}$, read λ off any row with a nonzero product; otherwise solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and back-substitute. For a symmetric $\mathbf{A}$, eigenvectors for distinct eigenvalues are orthogonal, and an orthogonal matrix $\mathbf{P}$ of normalised eigenvectors **diagonalises** it: $\mathbf{P}^T\mathbf{AP} = \mathbf{D}$.

Worked example – eigenvalues from a given eigenvector (WFM03 June 2025 Q3). $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ has eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Row 1: $1 - 2 + 4 = 3 = 3(1)$, so $\lambda = 3$. Row 2: $3+2a+b$ must equal $\lambda \times 2 = 6$, and row 3: $a+2+b$ must equal $\lambda \times 1 = 3$. So $2a+b = 3$ and $a+b = 1$: subtracting, $a = 2$ and $b = -1$. With $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$:

the trace is 2 and, expanding along row 1, $\det \mathbf{M} = 1(-2+1)+1(-3+2)+4(3-4) = -1-1-4 = -6$. The remaining eigenvalues then sum to $2-3=-1$ and multiply to $\frac{-6}{3} = -2$: they are 1 and -2. Eigenvectors: for $\lambda = 1$, solving $(\mathbf{M}-\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $-y+4z=0$ and $3x+y-z=0$, so $\mathbf{v} = (-1,4,1)$; for $\lambda=-2$, $(\mathbf{M}+2\mathbf{I})\mathbf{v}=\mathbf{0}$ gives $3x-y+4z=0$ and $2x+y+z=0$, so $\mathbf{v}=(-1,1,1)$ (both row-checked). Trace and determinant are the free verification: $3+1-2=2$ ✓ and $3\cdot 1\cdot(-2)=-6$ ✓.

Worked example – singular and inverse with a parameter (WFM03 June 2025 Q6). $\mathbf{A} = \begin{pmatrix} 1 & k & 2 \\ 5 & 3 & -2 \\ 6 & -1 & 4 \end{pmatrix}$: $\det \mathbf{A} = 1(12-2) - k(20+12) + 2(-5-18) = -36-32k$, so $\mathbf{A}$ is singular when $k = -\frac{9}{8}$. The adjugate (transposed cofactors) gives

$$\mathbf{A}^{-1} = \frac{1}{-36-32k}\begin{pmatrix} 10 & -4k-2 & -2k-6 \\ -32 & -8 & 12 \\ -23 & 6k+1 & 3-5k \end{pmatrix}.$$

Sanity-check the (1,1) entry of the product: $(1,k,2)\cdot(10,-32,-23) = 10-32k-46 = -36-32k$, which cancels against the determinant to give 1 ✓.

Checklist before you call this unit done

- Define $\cosh, \sinh, \tanh$ from exponentials; derive the core identities; solve mixed equations via e^x quadratics with exact ln answers.
- Derive the $\operatorname{arsinh}, \operatorname{arcosh}, \operatorname{artanh}$ log forms; differentiate inverses with domains respected.
- Use ellipse and hyperbola parametrics; convert between $b^2 = a^2(1-e^2)$ and $a^2(e^2-1)$; place foci and directrices; find tangents/normals at parametric points and via the $y = mx + c$ condition $c^2 = a^2m^2 \pm b^2$.
- Differentiate and integrate hyperbolic and inverse functions, including by parts for $\int$ arcosh-type integrands.
- Know the four surd/rational standard integrals; build reduction formulae by parts and iterate; compute arc lengths and surfaces of revolution exactly.
- Compute $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ with their area and volume meanings; move between plane and line forms; find point-to-plane and skew-line distances.
- Evaluate 3×3 determinants, invert via the adjugate, handle the singular case, find eigenvalues/eigenvectors (normalised when asked) and diagonalise symmetric matrices with an orthogonal $\mathbf{P}$.