

Further Pure Mathematics 1

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 1: complex numbers, matrices and proof

Further Pure Mathematics 1 (FP1, paper WFM01/01, 1 hour 30 minutes, 75 marks) is compulsory for International AS and A Level Further Mathematics and for International AS and A Level Pure Mathematics. P1–P2 are assumed and may be tested, including sign-change location of roots, rotating shapes about the origin, and dividing a cubic or quartic by a quadratic. This reference covers complex numbers, roots of quadratic equations, numerical methods, the parabola and rectangular hyperbola, matrix algebra, transformations, finite series and proof by induction.

Must-know for FP1 (not in the booklet):

Family	Formulae
quadratic roots	for $ax^2 + bx + c = 0$ with roots α, β : $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$
series	$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$
complex numbers	$z = a + bi$, $ z = \sqrt{a^2 + b^2}$, $\bar{z} = a - bi$, $ z_1 z_2 = z_1 z_2 $
conics	parabola $y^2 = 4ax$ (parametric $x = at^2$, $y = 2at$); rectangular hyperbola $xy = c^2$ (parametric $x = ct$, $y = \frac{c}{t}$)
matrices	$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, $\mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

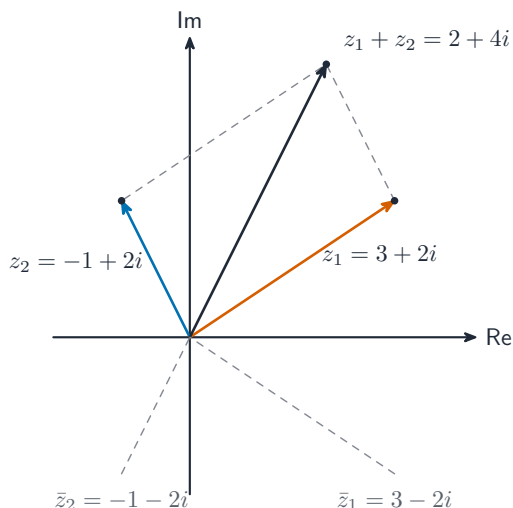
Complex numbers

A **complex number** is $z = a + bi$ with $i^2 = -1$; a is the **real part**, b the **imaginary part**. The **conjugate** $\bar{z} = a - bi$ reverses the sign of the imaginary part, and $z\bar{z} = a^2 + b^2 = |z|^2$ – the trick that makes a quotient real: $\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$. Two complex numbers are equal only when both real and imaginary parts match. The polar form $z = r(\cos \theta + i \sin \theta)$ pairs the **modulus** r with the **argument** θ ; multiplying complex numbers multiplies their moduli ($|z_1 z_2| = |z_1||z_2|$).

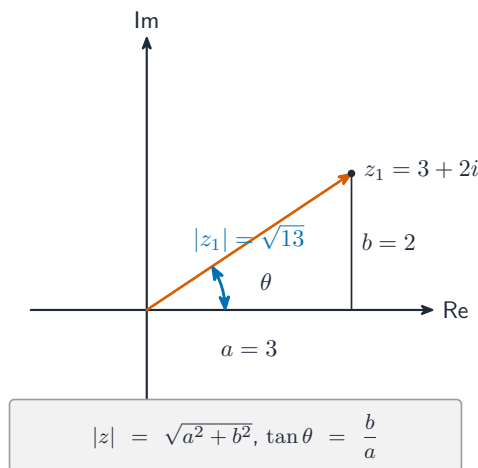
Worked example – a real sum pins the imaginary part. $z_1 = r(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6})$ with $r > 0$, and z_2 has modulus 5. Since $|z_1 z_2| = |z_1||z_2| = 5r = 15$, we get $r = 3$. So $z_1 = 3(-\frac{\sqrt{3}}{2} - \frac{1}{2}i) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$. If $z_1 + z_2$ is real, its imaginary part is zero, so

$z_2 = a + \frac{3}{2}i$; then $a^2 + \frac{9}{4} = 25$ gives $a = \pm \frac{\sqrt{91}}{2} \approx \pm 4.77$. Both $z_2 = \frac{\sqrt{91}}{2} + \frac{3}{2}i$ and its reflection in the imaginary axis work.

On an **Argand diagram** (Argand diagram) plot $z = a + bi$ as the point (a, b) : addition is vector addition, the conjugate reflects in the real axis, and $|z|$ is the distance from the origin.



conjugates reflect in the Re axis



Complex roots of polynomials

For a quadratic with real coefficients and negative discriminant, complete the square: $x^2 - 6x + 13 = (x - 3)^2 + 4 = 0$ gives $x = 3 \pm 2i$. The roots are a conjugate pair – and this survives degree: **if $f(z_1) = 0$ for real coefficients, then $f(\bar{z}_1) = 0$ too.** So a real cubic has either three real roots or one real root plus one conjugate pair, and $(z - z_1)(z - \bar{z}_1) = z^2 - 2\operatorname{Re}(z_1)z + |z_1|^2$ is a real quadratic factor. Divide it out to reach the rest.

Worked example – a quartic with a given complex root (WFM01 June 2025 Q7). $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3 + 5i$, so $3 - 5i$ is also a root and

$$z^2 - 6z + 34 \text{ is a factor} \quad (\text{sum } 6, \text{ product } 9 + 25 = 34).$$

Writing $f(z) = (z^2 - 6z + 34)(Pz^2 + bz + c)$ and comparing: $34c = 68$ so $c = 2$; $-6c + 34b = 192$ so $34b = 204$, $b = 6$; then $b - 6P = -36$ gives $P = 7$ and $c - 6b + 34P = Q$ gives $Q = 204$. (Check against $f(3 + 5i) = 0$: the real equation $644P + 16Q + 7772$ -type balance holds since $644(7) + 16(204) - 7772 = 0$.) The other roots solve $7z^2 + 6z + 2 = 0$:

$$z = \frac{-6 \pm \sqrt{36 - 56}}{14} = \frac{-6 \pm 2i\sqrt{5}}{14} = \frac{-3 \pm i\sqrt{5}}{7}.$$

Without a calculator: only $b^2 - 4ac < 0$ arithmetic and the conjugate fact are used.

Roots of quadratic equations

For $ax^2 + bx + c = 0$ with roots α, β : $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Symmetric expressions in α, β then need no individual roots: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$, $\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$, and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$. To build a new quadratic, express the new **sum** and **product** in terms of the old ones; the equation is $x^2 - (\text{sum})x + \text{product} = 0$.

Worked example – new roots without solving (WFM01 June 2025 Q6).

$f(x) = 3x^2 + kx - 5$ has roots α, β , so $\alpha\beta = -\frac{5}{3}$. If $\alpha + \beta = 9\alpha\beta$, then $-\frac{k}{3} = -15$, so $k = 45$ and $\alpha + \beta = -15$. For roots $\alpha^2 + \beta$ and $\alpha + \beta^2$:

$$\text{sum} = \alpha^2 + \beta^2 + \alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta + (\alpha + \beta) = 225 + \frac{10}{3} - 15 = \frac{640}{3},$$

$$\text{product} = (\alpha^3 + \beta^3) + \alpha\beta + (\alpha\beta)^2 = -3450 - \frac{5}{3} + \frac{25}{9} = -\frac{31040}{9}.$$

The new equation is $9x^2 - 1920x - 31040 = 0$ (multiply $x^2 - \frac{640}{3}x - \frac{31040}{9} = 0$ by 9).

Numerical solution of equations

Three iterative locators, all needing f continuous on the interval: **interval bisection** halves an interval carrying a sign change; **linear interpolation**

chords from $(a, f(a))$ to $(b, f(b))$ and takes the chord's zero $\beta \approx a - \frac{(b-a)f(a)}{f(b) - f(a)}$; the

Newton–Raphson method (formula given in the booklet) iterates

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Show every evaluation – “calculator-only” answers score nothing.

Worked example – bisection and interpolation (WFM01 June 2025 Q3).

(i) $f(x) = x^2 + 5 - 8^{5x}$ has a single root α in $[0, 1]$: $f(0) = 4 > 0$, $f(0.5) = 5.25 - 8^{2.5} = -175.77 < 0$, so halve: $f(0.25) = 5.0625 - 8^{1.25} = -8.392 < 0$. The sign change sits in $[0, 0.25]$. (ii) $g(x) = 3^{\sin x} - 3 \cos x$ (radians): $g(4) = 2.396 > 0$ and $g(5) = -0.502 < 0$, and g is continuous, so a root β lies in $[4, 5]$. Interpolating once:

$$\beta \approx 4 + \frac{2.396}{2.396 + 0.502} = 4.827 \text{ (4 s.f.)}.$$

Newton–Raphson (original example). Solve $x^3 - 2x - 5 = 0$ from $x_0 = 2$: $f(2) = -1$, $f'(x) = 3x^2 - 2$ so $f'(2) = 10$ and $x_1 = 2 - \frac{-1}{10} = 2.1$; then $f(2.1) = 0.061$, $f'(2.1) = 11.23$, $x_2 = 2.0946$. The iterates home in on the true root 2.0946... (check: $2.0946^3 - 2(2.0946) - 5 \approx +0.0005$).

Parabola and rectangular hyperbola

The **parabola** $y^2 = 4ax$ has parametric form $x = at^2$, $y = 2at$; the **rectangular hyperbola** $xy = c^2$ has $x = ct$, $y = \frac{c}{t}$. Tangents and normals come from differentiating the Cartesian equation (parametric differentiation is not required): $y^2 = 4ax$ gives $\frac{dy}{dx} = \frac{2a}{y} = \frac{1}{t}$ at parameter t , and $xy = c^2$ gives $y = \frac{c^2}{x}$ so $\frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{1}{t^2}$ at $x = ct$.

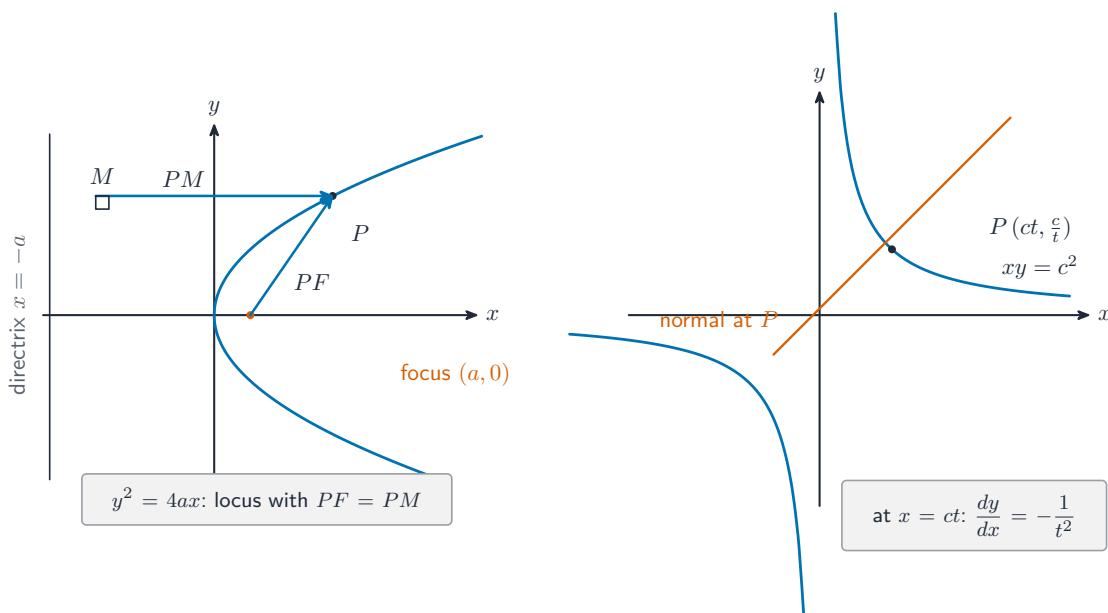
Worked example – normal to a hyperbola (WFM01 June 2025 Q4). For $xy = 81$ at $P(9t, \frac{9}{t})$: $\frac{dy}{dx} = -\frac{81}{x^2} = -\frac{1}{t^2}$, so the normal has gradient t^2 :

$$y - \frac{9}{t} = t^2(x - 9t) \implies ty = t^3x + 9(1 - t^4).$$

At the y -axis, $A = (0, \frac{9(1-t^4)}{t})$. With $t = \frac{1}{3}$: $P = (3, 27)$ and $A = (0, \frac{80}{3})$, so triangle OPA has area $\frac{1}{2} \times 3 \times \frac{80}{3} = 40$.

The parabola is the locus of points equidistant from the **focus** $(a, 0)$ and the **directrix** $x = -a$. Distances, not algebra, unlock the geometry questions.

Worked example – focus-directrix distances (WFM01 June 2025 Q9). P lies on the parabola 5 units from the focus, so P is also 5 units from the directrix ($PF = PM = 5$, with PM perpendicular to the directrix). Points Q, R on the directrix satisfy $PQ = PR = 10$. Each right triangle PQM gives $QM = \sqrt{10^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$, and M lies between Q and R , so $QR = 10\sqrt{3}$.



Matrix algebra

Matrices add entry-wise (same shape), scale entry-wise, and multiply row-by-column ($(\mathbf{AB})_{ik} = \sum_j A_{ij}B_{jk}$; $\mathbf{AB} \neq \mathbf{BA}$ in general). For 2×2 matrices $\det \mathbf{M} = ad - bc$: zero means **singular** (no inverse); otherwise $\mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\mathbf{MM}^{-1} = \mathbf{I}$. Inverses reverse order: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ – undo the last transformation first.

Worked example – an inverse that must stay consistent (WFM01 June 2025

Q1). $\mathbf{M} = \begin{pmatrix} 1 & a \\ 3 & -5 \end{pmatrix}$ with $a \neq -\frac{5}{3}$:

$$\mathbf{M}^{-1} = \frac{1}{-(5 + 3a)} \begin{pmatrix} -5 & -a \\ -3 & 1 \end{pmatrix}.$$

Given $\mathbf{M}^{-1} = 2\mathbf{M} + 8\mathbf{I} = \begin{pmatrix} 10 & 2a \\ 6 & -2 \end{pmatrix}$, compare entries: $\frac{-3}{-(5 + 3a)} = 6$ gives $5 + 3a = \frac{1}{2}$, and every other entry confirms it, so $a = -\frac{3}{2}$.

Transformations using matrices

A 2×2 matrix $\mathbf{M}$ transforms column vectors by $\mathbf{y} = \mathbf{Mx}$; its **columns are the images of $\mathbf{i}$ and $\mathbf{j}$** – read any transformation straight off the matrix. The standard results:

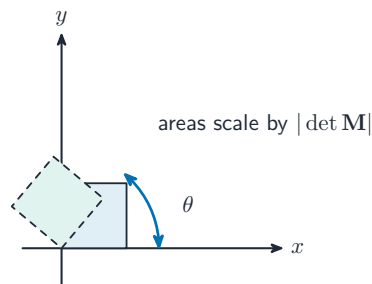
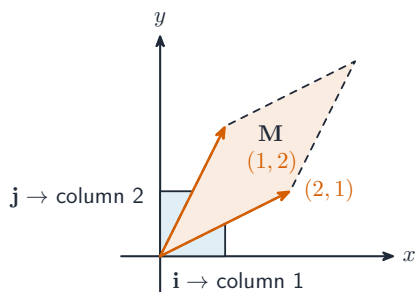
Transformation	Matrix
reflection in the x -axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection in the y -axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
reflection in $y = x$ / $y = -x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} / \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
rotation, angle θ anticlockwise about O	$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$
stretch, factor k parallel to the x -axis	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ (swap for the y -axis)
enlargement, scale factor k about O	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

For combinations, $\mathbf{AB}$ means "apply $\mathbf{B}$ first, then $\mathbf{A}$ ". The inverse transformation undoes it, and $|\det \mathbf{M}|$ is the **area scale factor** : areas multiply by the

determinant (negative determinants add a reflection).

Worked example – describing and combining (WFM01 June 2025 Q2).

$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ maps $\mathbf{i} \rightarrow (0, -1)^T$ and $\mathbf{j} \rightarrow (1, 0)^T$: a quarter-turn clockwise (270° anticlockwise) about O . $\mathbf{Q} = 5\mathbf{I}$ is an enlargement, scale factor 5, centre O . Reflection in $y = -x$ is $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, so $A = \mathbf{R}(4, 3)^T = (-3, -4)$. If $\mathbf{Q}$ follows $\mathbf{P}$ and maps B to A , then $B = \mathbf{P}^{-1}\mathbf{Q}^{-1}A = \mathbf{P}^{-1} \cdot \frac{1}{5}(-3, -4)^T = \frac{1}{5}(4, -3)^T = (0.8, -0.6)$ – $\mathbf{P}$'s own inverse is its transpose, so this is arithmetic only.



rotation by θ : $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Series

With $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$ (and $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$ from the booklet), any polynomial summand splits into standard pieces – including sums to an even limit $2n$ (substitute $k = 2n$ into the standard formulae) and shifted sums like $\sum (r+1)^2$.

Worked example – a sum to $2n$ (WFM01 June 2025 Q8a). Show $\sum_{r=1}^{2n} (2r^2 - 1) = \frac{4}{3}n(n+1)(an+b)$:

$$2 \sum_{r=1}^{2n} r^2 - 2n = 2 \cdot \frac{(2n)(2n+1)(4n+1)}{6} - 2n = \frac{2n(2n+1)(4n+1) - 6n}{3} = \frac{2n(8n^2 + 6n - 2)}{3} = \frac{4n(4n^2 + 3n - 1)}{3},$$

and $4n^2 + 3n - 1 = (n+1)(4n-1)$, so the sum is $\frac{4}{3}n(n+1)(4n-1)$: $a = 4$, $b = -1$. (Check $n = 1$: $1 + 7 = 8$ and $\frac{4}{3} \cdot 1 \cdot 2 \cdot 3 = 8$.)

Proof by mathematical induction

Induction proves a statement $P(n)$ for all positive integers in three moves: verify $P(1)$; assume $P(k)$; deduce $P(k+1)$ from it – then close with "true for $n = 1$, and truth for $n = k$ implies truth for $n = k+1$, so by induction true for all

positive integers n ". The closing sentence is part of the proof. Four assessed families: summation formulae, divisibility, sequences given by a recurrence, and matrix powers.

Worked example – a summation (WFM01 June 2025 Q8b). Prove $\sum_{r=1}^n r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$. For $n = 1$: LHS = $1 \cdot 1^2 = 1$ and RHS = $\frac{1}{4} \cdot 1 \cdot 2 \cdot 2 = 1$ – true. Assume true for $n = k$; the $(k+1)$ th term is $(k+1)(3k+1)^2$, so

$$\sum_{r=1}^{k+1} r(3r-2)^2 = \frac{1}{4}k^2(k+1)(9k-7) + (k+1)(3k+1)^2 = \frac{1}{4}(k+1)[k^2(9k-7) + 4(3k+1)^2].$$

Expand the bracket: $9k^3 - 7k^2 + 36k^2 + 24k + 4 = 9k^3 + 29k^2 + 24k + 4$. The target for $n = k+1$ is $\frac{1}{4}(k+1)^2(k+2)(9k+2)$, and its bracket $(k+1)(k+2)(9k+2)$ expands as $(k^2 + 3k + 2)(9k + 2) = 9k^3 + 29k^2 + 24k + 4$ – identical. So the statement holds for $n = k+1$, and by induction for all positive integers n .

Divisibility (original example). Prove $n^3 + 2n$ is divisible by 3. For $n = 1$: 3. Assume $k^3 + 2k = 3m$; then $(k+1)^3 + 2(k+1) = (k^3 + 2k) + 3k^2 + 3k + 3 = 3(m + k^2 + k + 1)$, still a multiple of 3. By induction, true for all n .

Matrix powers (original example). Prove $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. True for $n =$

1. If true for k , then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$, completing the induction.

Checklist before you call this unit done

- Work with $a + bi$ and $r(\cos \theta + i \sin \theta)$: modulus, argument, conjugate; quotients made real.
- Use conjugate pairs to factor cubics and quartics with real coefficients; find all roots without a calculator.
- Convert symmetric expressions via $\alpha + \beta$ and $\alpha\beta$; form new quadratics for transformed roots.
- Locate roots by bisection and linear interpolation with every evaluation shown; iterate Newton–Raphson from a given start.
- Use $y^2 = 4ax$ and $xy = c^2$ parametrics; tangents/normals via Cartesian differentiation; focus-directrix distances.
- Invert 2×2 matrices (and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$); solve matrix equations entry by entry.
- Identify reflections, rotations, stretches, enlargements from matrices; combine in the right order; use $|\det|$ as the area scale factor.
- Sum series to n and $2n$ with the standard formulae; prove series, divisibility, recurrence and matrix results by induction with a closing statement.