

## 2 • Pure Mathematics 2

Pearson Edexcel • International A-Level • Pure Mathematics

### Pure Mathematics 2: building on the P1 toolkit

Pure Mathematics 2 (P2) is the second compulsory unit of International AS Mathematics and International AS Pure Mathematics, and of both International A Level routes. This reference covers every P2 topic in the Edexcel IAL specification (Issue 3, April 2019): proof, the factor and remainder theorems, circle geometry, sequences and series, exponentials and logarithms, trigonometric identities and equations, stationary points, and integration including areas and the trapezium rule.

- The P2 paper (WMA12/01) lasts **1 hour 30 minutes** with **75 marks**. A knowledge of P1 and its formulae is assumed and **may be tested** on the P2 paper.
- The same calculator and yellow *Mathematical Formulae and Statistical Tables* rules apply as for P1. The formulae below do **not** appear in the booklet – memorise them.
- “Show all stages of your working” questions reject calculator-only answers; give inexact answers to 3 significant figures unless told otherwise.

**Formulae you must know for P2** (from the specification’s assessment information):

| Formula                                                                                                                    | Use                          |
|----------------------------------------------------------------------------------------------------------------------------|------------------------------|
| $\log_a x + \log_a y \equiv \log_a(xy), \log_a x - \log_a y \equiv \log_a \frac{x}{y},$<br>$k \log_a x \equiv \log_a(x^k)$ | laws of logarithms           |
| $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}, \sin^2 \theta + \cos^2 \theta \equiv 1$                               | trigonometric identities     |
| area under a curve $= \int_a^b y \, dx \quad (y \geq 0)$                                                                   | definite integral as an area |

### Proof

A **proof** proceeds from given assumptions through logical steps to a conclusion. P2 uses three methods:

- **Proof by exhaustion** : split the possibilities into cases and check every case.
- **Disproof by counterexample** : one example that breaks the claim disproves it.

- Algebraic proof: start from what is given, manipulate identities (not equations to solve), and end with the target statement.

**Worked example – exhaustion** (adapted from Edexcel IAL WMA12 June 2025 Q10). Prove that for all positive integers  $m$  that are not multiples of 3, the value of  $m^2 + 3m + 2$  is always a multiple of 3.

- Factorise first:  $m^2 + 3m + 2 = (m + 1)(m + 2)$ .
- A positive integer not divisible by 3 is either  $m = 3q + 1$  or  $m = 3q + 2$  – those are all the cases, so exhausting them proves the claim.
- Case  $m = 3q + 1$ :  $(m + 1)(m + 2) = (3q + 2)(3q + 3) = 3(3q + 2)(q + 1)$  – a multiple of 3.
- Case  $m = 3q + 2$ :  $(m + 1)(m + 2) = (3q + 3)(3q + 4) = 3(q + 1)(3q + 4)$  – a multiple of 3.
- Both cases give a multiple of 3, so  $3 \mid m^2 + 3m + 2$  whenever  $3 \nmid m$ . ■
- Check:  $m = 1$ :  $6 = 3 \times 2$  ✓;  $m = 2$ :  $12 = 3 \times 4$  ✓;  $m = 4$ :  $30$  ✓;  $m = 5$ :  $42$  ✓.

**Worked example – counterexample.** Disprove: “ $n^2 - n + 1$  is prime for all positive integers  $n$ .”

- One counterexample is enough. Try  $n = 5$ :  $25 - 5 + 1 = 21 = 3 \times 7$ , which is not prime.
- The claim is false. Check  $n = 1, 2, 3, 4$  give 1, 3, 7, 13 – the pattern holds at first, which is exactly why a counterexample, not more examples, settles the question.

### Algebra: the factor and remainder theorems

For a polynomial  $f(x)$ :

- **Factor theorem** :  $(x - b)$  is a factor of  $f(x)$  exactly when  $f(b) = 0$ . For divisors of the form  $(ax - b)$ : if  $f(\frac{b}{a}) = 0$  then  $(ax - b)$  is a factor.
- **Remainder theorem** : dividing  $f(x)$  by  $(ax - b)$  leaves remainder  $f(\frac{b}{a})$ .
- **Algebraic division** by  $(ax \pm b)$  gives a **quotient** and a **remainder**; the degree drops by one:  $f(x) = (ax - b) \times (\text{quotient}) + \text{remainder}$ .

**Worked example – both theorems at once** (adapted from Edexcel IAL WMA12 June 2025 Q4).  $f(x) = ax^3 + bx^2 + 5x - 3$ , where  $(x + 3)$  is a factor of  $f(x)$  and dividing  $f(x)$  by  $(2x - 1)$  leaves remainder 7.

(a) Show that  $b = 3a + 2$ .

- Factor theorem:  $f(-3) = 0$ :

$$a(-27) + b(9) + 5(-3) - 3 = 0 \implies -27a + 9b - 18 = 0 \implies b = 3a + 2$$

(b) Find  $a$  and  $b$ .

- Remainder theorem with  $(2x - 1)$ :  $f(\frac{1}{2}) = 7$ :

$$\frac{a}{8} + \frac{b}{4} + \frac{5}{2} - 3 = 7 \implies \frac{a}{8} + \frac{b}{4} = 7.5 \implies a + 2b = 60$$

- Substitute  $b = 3a + 2$ :  $a + 6a + 4 = 60$ , so  $a = 8$  and  $b = 26$ .
- Check:  $f(x) = 8x^3 + 26x^2 + 5x - 3$ ;  $f(-3) = -216 + 234 - 15 - 3 = 0$  ✓ and  $f(\frac{1}{2}) = 1 + 6.5 + 2.5 - 3 = 7$  ✓.

(c) Quotient and remainder when  $f(x)$  is divided by  $(x - 2)$ .

- Divide  $8x^3 + 26x^2 + 5x - 3$  by  $(x - 2)$ : quotient  $8x^2 + 42x + 89$ , remainder 175.
- Check by rebuilding:  $(x - 2)(8x^2 + 42x + 89) + 175 = 8x^3 + 42x^2 + 89x - 16x^2 - 84x - 178 + 175 = 8x^3 + 26x^2 + 5x - 3$  ✓. The remainder also equals  $f(2) = 64 + 104 + 10 - 3 = 175$  ✓.

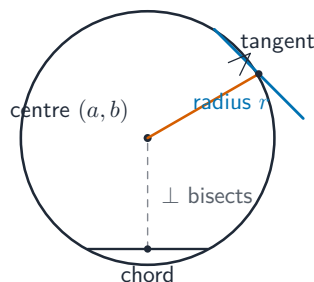
## Coordinate geometry of the circle

A circle centre  $(a, b)$  radius  $r$  has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Given the expanded form  $x^2 + y^2 + fx + gy + c = 0$ , complete the square to recover centre  $(-\frac{f}{2}, -\frac{g}{2})$  and  $r^2 = \frac{f^2 + g^2}{4} - c$ . Three circle properties are examined:

- the angle in a semicircle is a right angle;
- the perpendicular from the centre to a **chord** bisects the chord;
- a **tangent** is perpendicular to the radius at the point of contact.



Tangent  $\perp$  radius at the contact point; the perpendicular from the centre to a chord **bisects** it; the angle in a semicircle is 90.

*Radius  $\perp$  tangent; centre-to-chord perpendicular bisects the chord.*

**Worked example – diameter endpoints** (adapted from Edexcel IAL WMA12 June 2025 Q2). The line joining  $(-2, 5)$  and  $(4, 15)$  is a diameter of circle  $C$ .

(a) Equation of  $C$ .

- The centre is the midpoint:  $(\frac{-2+4}{2}, \frac{5+15}{2}) = (1, 10)$ .
- The diameter squared:  $d^2 = (4 - (-2))^2 + (15 - 5)^2 = 36 + 100 = 136$ , so  $r^2 = 136 \div 4 = 34$ .

$$(x - 1)^2 + (y - 10)^2 = 34$$

- Check: both endpoints satisfy the equation:  $(-3)^2 + (-5)^2 = 34$  ✓ and  $3^2 + 5^2 = 34$  ✓.

(b) Exact coordinates of the point on  $C$  nearest the  $x$ -axis.

- The nearest point lies on the vertical radius: same  $x$  as the centre,  $r$  below it:

$$(1, 10 - \sqrt{34})$$

- Check:  $(1 - 1)^2 + (10 - \sqrt{34} - 10)^2 = 34$  ✓, and no smaller  $y$  is possible since the centre sits  $10 > \sqrt{34}$  above the axis.

## Sequences and series

A **sequence** may be given by a formula for the  $n$ th term or by a relation  $x_{n+1} = f(x_n)$ . Describe it as **increasing**, **decreasing** or **periodic** (from its terms, not its formula).

**Arithmetic sequences** : constant difference  $d$ .

$$a_n = a + (n - 1)d \quad S_n = \frac{n}{2}(2a + (n - 1)d) \quad \sum_{r=1}^n r = \frac{n(n + 1)}{2}$$

The proof of the sum formula (write  $S_n$  forwards and backwards, add) should be known, as should  $\Sigma$  notation.

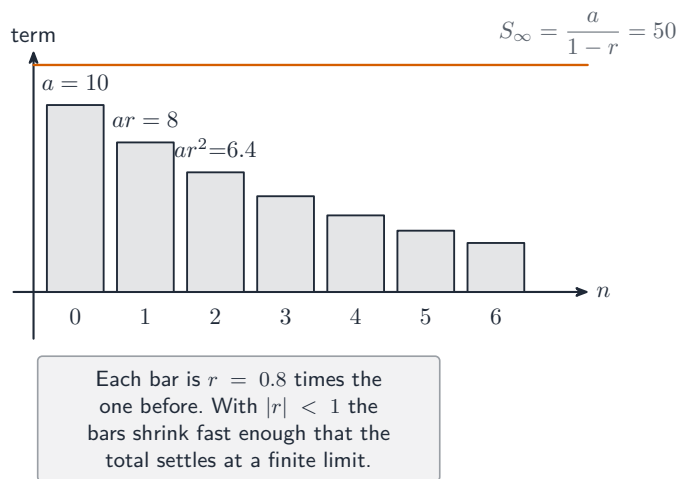
**Geometric sequences** : constant ratio  $r$ .

$$a_n = ar^{n-1} \quad S_n = \frac{a(1 - r^n)}{1 - r} \quad S_\infty = \frac{a}{1 - r} \quad (|r| < 1)$$

The sum formula's proof (telescoping  $S_n - rS_n$ ) should be known. Use **logarithms** to solve for  $n$  when a term value is given.

**Binomial expansion** for a positive integer  $n$ :

$$(a + bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \cdots + (bx)^n, \quad \binom{n}{r} = \frac{n!}{r!(n - r)!}$$



$|r| < 1$  means the terms shrink; the sum settles at  $S_{\infty}$ .

**Worked example – geometric series with logs** (adapted from Edexcel IAL WMA12 June 2025 Q7). A geometric series begins  $10 + 8 + 6.4 + \dots$

(a) Sum to infinity.

- $a = 10$ ,  $r = 8/10 = 0.8$ , and  $|r| < 1$ :

$$S_{\infty} = \frac{a}{1-r} = \frac{10}{1-0.8} = 50$$

(b) The  $k$ th term is less than 0.0005 – find the smallest  $k$ , by algebra.

- The  $k$ th term is  $10(0.8)^{k-1}$ :

$$10(0.8)^{k-1} < 0.0005 \implies (0.8)^{k-1} < 0.00005$$

- Take logs (the log of a number below 1 is negative – keep the inequality direction when dividing by it):

$$(k-1) \log 0.8 < \log 0.00005 \implies k-1 > \frac{\log 0.00005}{\log 0.8} = 44.38 \dots$$

- So  $k > 45.38$  and the smallest integer is  $k = 46$ .
- Check:  $10(0.8)^{45} = 0.00045 \dots < 0.0005$  ✓ but  $10(0.8)^{44} = 0.00056 > 0.0005$  ✓.

**Worked example – maximum arithmetic sum** (adapted from the same question). An arithmetic series begins  $850 + 843 + 836 + \dots$  ( $d = -7$ ). Its greatest partial sum:

- Terms stay positive until  $u_n = 850 - 7(n-1) > 0$ , so  $n \leq 122.4$ ; the 122nd term is  $850 - 7(121) = 3 > 0$  and the 123rd is  $-4 < 0$ .

$$S_{122} = \frac{122}{2}(2(850) + 121(-7)) = 61(1700 - 847) = 61 \times 853 = 52\,033$$

- Check: adding a negative 123rd term would reduce the sum, so  $S_{122}$  is the maximum ✓.

## Exponentials and logarithms

$y = a^x$  ( $a > 0$ ,  $a \neq 1$ ) passes through  $(0, 1)$  and grows without bound for  $a > 1$ . The **logarithm**  $\log_a x$  inverts it:  $\log_a x = y$  means  $a^y = x$ . Know:

$$\log_a x + \log_a y \equiv \log_a(xy) \quad \log_a x - \log_a y \equiv \log_a \frac{x}{y} \quad k \log_a x \equiv \log_a(x^k) \quad \log_a a = 1$$

and  $\log_a \frac{1}{x} = -\log_a x$ . Solve  $a^x = b$  with  $x = \log_a b$  (a change of base may be used). Always check candidates against the **domain**: logs need positive arguments.

**Worked example – a log equation** (adapted from Edexcel IAL WMA12 June 2025 Q6). Solve  $2 \log_2(4-x) = 3 + \log_2\left(\frac{x+11}{2}\right)$ .

- Bring everything to one log per side:

$$\log_2(4-x)^2 = \log_2 8 + \log_2\left(\frac{x+11}{2}\right) = \log_2(4(x+11))$$

- Equal logs give equal arguments:

$$(4-x)^2 = 4(x+11) \implies x^2 - 8x + 16 = 4x + 44 \implies x^2 - 12x - 28 = 0$$

$$(x-14)(x+2) = 0 \implies x = 14 \text{ or } x = -2$$

- Domain check:  $4-x > 0$  needs  $x < 4$ , so  $x = 14$  is rejected.  $x = -2$ .
- Check: LHS =  $2 \log_2 6$ ; RHS =  $3 + \log_2(4.5) = \log_2(8 \times 4.5) = \log_2 36 = 2 \log_2 6$  ✓.

**Worked example – exact intersection in  $\log_3$  form** (same question). The curves  $y = 3^{2x+1}$  and  $y = 6 \times 3^x$  meet at  $P$ ; write  $P$  as  $(\log_3 a, b)$ .

- Set equal and substitute  $u = 3^x$ :

$$3 \cdot 3^{2x} = 6 \cdot 3^x \implies 3u^2 = 6u \implies u = 2 \implies 3^x = 2 \implies x = \log_3 2$$

$$y = 6 \times 3^x = 6 \times 2 = 12$$

- $P = (\log_3 2, 12)$ , so  $a = 2$ ,  $b = 12$ . Check:  $3^{2 \log_3 2 + 1} = 3 \cdot 3^{2 \log_3 2} = 3 \cdot 4 = 12$  ✓.

## Trigonometric identities and equations

Two identities convert between sine, cosine and tangent:

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta} \quad \sin^2 \theta + \cos^2 \theta \equiv 1$$

Solve **trigonometric equations** in a given interval by reducing to one function (often a hidden quadratic), then reading the second solution from the graph's symmetry. State the interval, work in its units (degrees or radians), and reject values outside.

**Worked example – radians and exact fractions** (adapted from Edexcel IAL WMA12 June 2025 Q8). Solve  $3 \tan\left(2x + \frac{\pi}{5}\right) = \sqrt{3}$  for  $0 \leq x < \pi$ .

- Divide by 3:  $\tan\left(2x + \frac{\pi}{5}\right) = \frac{1}{\sqrt{3}}$ , and  $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ :

$$2x + \frac{\pi}{5} = \frac{\pi}{6} + k\pi \implies 2x = -\frac{\pi}{30} + k\pi \implies x = -\frac{\pi}{60} + \frac{k\pi}{2}$$

- In  $0 \leq x < \pi$ :  $x = \frac{29\pi}{60}$  ( $k = 1$ ) and  $x = \frac{59\pi}{60}$  ( $k = 2$ ).
- Check: at  $x = \frac{29\pi}{60}$ :  $2x + \frac{\pi}{5} = \frac{7\pi}{6}$  and  $\tan \frac{7\pi}{6} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$  ✓.

**Worked example – hidden quadratic in cosine** (same question). Solve  $5 \sin \theta \tan \theta = \cos \theta + 4$  for  $0 \leq \theta < 360^\circ$ .

- Replace  $\tan \theta = \sin \theta / \cos \theta$  and multiply through by  $\cos \theta$ :

$$\frac{5 \sin^2 \theta}{\cos \theta} = \cos \theta + 4 \implies 5 \sin^2 \theta = \cos^2 \theta + 4 \cos \theta$$

- Use  $\sin^2 \theta = 1 - \cos^2 \theta$ :

$$5(1 - \cos^2 \theta) = \cos^2 \theta + 4 \cos \theta \implies 6 \cos^2 \theta + 4 \cos \theta - 5 = 0$$

- Quadratic formula in  $\cos \theta$ :

$$\cos \theta = \frac{-4 \pm \sqrt{16 + 120}}{12} = \frac{-2 \pm \sqrt{34}}{6}$$

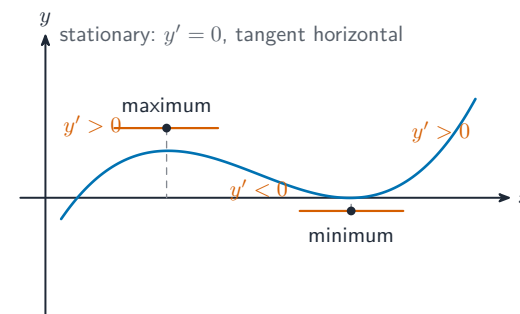
- The  $-$  root is below  $-1$  (reject);  $\cos \theta = 0.6385$  gives  $\theta = 50.3^\circ$  and, from cosine's symmetry,  $\theta = 360^\circ - 50.3^\circ = 309.7^\circ$ .
- Check:  $\cos 50.3^\circ = 0.638$  ✓; both angles lie in the interval ✓.

## Differentiation: stationary points

**Stationary points** have  $\frac{dy}{dx} = 0$ . Classify with the sign of the derivative:

- derivative changes  $+$  to  $-$ : **maximum** ;
- derivative changes  $-$  to  $+$ : **minimum** ;
- the second derivative helps:  $f''(x) > 0$  at the point confirms a minimum,  $f''(x) < 0$  a maximum.

A function is **increasing** where  $f'(x) > 0$  and **decreasing** where  $f'(x) < 0$ . Practical problems – least material, least cost – translate into minimising an expression in one variable.



*Stationary means the tangent is horizontal; the sign change classifies it.*

**Worked example – a modelling minimum** (adapted from Edexcel IAL WMA12 June 2025 Q9). An open trough has right-triangle ends with legs  $3x$  and  $2x$  (metres), rectangular sides  $3x \times L$  and  $2x \times L$ , and capacity  $12 \text{ m}^3$ . The metal area is  $S \text{ m}^2$ .

(a) Show  $S = 6x^2 + \frac{20}{x}$ .

- Volume: cross-section  $\times$  length  $= \frac{1}{2}(3x)(2x) \times L = 3x^2L = 12$ , so  $L = \frac{4}{x^2}$ .
- Surface: two triangles plus the two rectangles:

$$S = 2 \times \frac{1}{2}(3x)(2x) + 3xL + 2xL = 6x^2 + 5x \cdot \frac{4}{x^2} = 6x^2 + \frac{20}{x}$$

(b) Minimum  $S$  by calculus.

$$\frac{dS}{dx} = 12x - \frac{20}{x^2} = 0 \implies 12x^3 = 20 \implies x^3 = \frac{5}{3} \implies x = 1.1856 \dots$$

$$S = 6(1.1856)^2 + \frac{20}{1.1856} = 8.43 + 16.87 = 25.3 \text{ m}^2 \text{ (1 d.p.)}$$

(c) Justify the minimum.

$$\frac{d^2S}{dx^2} = 12 + \frac{40}{x^3} > 0 \quad \text{for all } x > 0$$

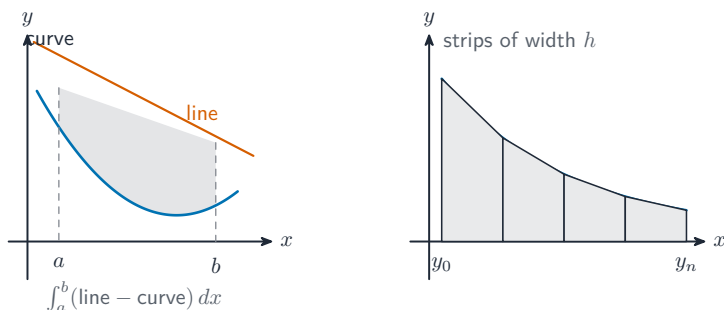
- Positive second derivative everywhere on the domain: the stationary point is a minimum ✓.
- Check: at  $x = 1.1856$ ,  $\frac{d^2S}{dx^2} = 12 + 40/(5/3) = 12 + 24 = 36 > 0$  ✓.

## Integration: areas and the trapezium rule

Evaluate **definite integrals** with the fundamental theorem:

$$\int_a^b y \, dx = F(b) - F(a) \quad \text{where } F'(x) = y$$

The definite integral is the **area under the curve** for  $y \geq 0$ . For a region bounded by a curve and a line, integrate (line – curve) between their intersections, or subtract the two areas – always draw a sketch to decide which way round.



*Area between =  $\int (\text{line} - \text{curve})$ ; the trapezium rule estimates  $\int$  curve with straight strips.*

**Worked example – area between curve and line** (adapted from Edexcel IAL WMA12 June 2025 Q3).  $C: y = 3x + \frac{16}{x^2} - 8$  ( $x > 0$ ) with  $P(1, 11)$  and  $Q(4, 5)$  on  $C$ ; find the area between  $C$  and the line segment  $PQ$ .

- Line  $PQ$ : gradient  $\frac{5-11}{4-1} = -2$ , so  $y - 11 = -2(x - 1) \implies y = -2x + 13$ .
- Check  $Q$ :  $-8 + 13 = 5$  ✓.
- Area under the line:  $\int_1^4 (-2x + 13) \, dx = [-x^2 + 13x]_1^4 = 36 - 12 = 24$ .
- Area under the curve:  $\int_1^4 (3x + 16x^{-2} - 8) \, dx = [\frac{3}{2}x^2 - \frac{16}{x} - 8x]_1^4 = (24 - 4 - 32) - (\frac{3}{2} - 16 - 8) = 10.5$ .
- Region area:  $24 - 10.5 = \mathbf{13.5}$  square units.
- Check: the line is above the curve throughout (at  $x = 2$ : line 9, curve  $3(2) + 4 - 8 = 2$ ), so subtracting curve from line is the right way round ✓.

## The trapezium rule

approximates an area with  $n$  strips of width  $h$ :

$$\int_a^b y \, dx \approx \frac{h}{2} \left[ \frac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1} \right] \quad \text{or} \quad \frac{h}{2} [y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$$

More strips (or a straighter curve) improve the estimate; a curve that bends upwards (**convex**, U-shaped like  $x^2$ ) leaves every chord **above** it, so the rule **overestimates**, while a curve that bends downwards (**concave**, like  $\sqrt{x}$  or a circle arc) leaves chords **below** it, so the rule **underestimates**.

**Worked example – trapezium rule with a table.** Estimate  $\int_1^3 \frac{1}{x} \, dx$  with 4 strips.

- Four strips over  $[1, 3]$  give  $h = 0.5$  and five values of  $y = \frac{1}{x}$ : at  $x = 1, 1.5, 2, 2.5, 3$  the values are 1, 0.6667, 0.5, 0.4, 0.3333.

$$\int_1^3 \frac{1}{x} \, dx \approx \frac{0.5}{2} [1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)] = 0.25 \times 4.4667 = 1.12$$

- The true value is  $\ln 3 = 1.099$  (3 s.f.). The estimate is an **overestimate**:  $y = 1/x$  is convex (it bends upwards), so every straight strip sits above the curve ✓.

## Checklist before you call this unit done

- Structure a proof; use exhaustion over complete cases; disprove with one counterexample.
- Apply the factor and remainder theorems, including  $(ax - b)$  divisors, and carry out algebraic division with quotient and remainder.
- Find a circle's centre and radius from either form; use the three circle properties.
- Use  $a_n$ ,  $S_n$  and  $S_\infty$  for arithmetic and geometric series (both sum-formula proofs),  $\Sigma$  notation, and logs to find  $n$ .
- Expand  $(a + bx)^n$  with binomial coefficients, and match given coefficients.
- Solve log equations with the three laws, checking domains; give intersections exactly as  $(\log_a b, c)$  when asked.
- Reduce trigonometric equations with  $\tan = \sin / \cos$  and  $\sin^2 + \cos^2 = 1$ ; solve in degrees or radians as given, listing all interval solutions.
- Find and classify stationary points; justify with a sign change or second derivative; set up modelling minima in one variable.
- Evaluate definite integrals; find areas under curves and between curve and line; apply the trapezium rule from a table and state the direction of its error.