

Solutions

2.1 $\mathbf{i}((-3)(-2) - (1)(4)) - \mathbf{j}((2)(-2) - (1)(1)) + \mathbf{k}((2)(4) - (-3)(1)) = 2\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}.$

2.2 $\mathbf{b} \times \mathbf{c} = (1 \cdot 1 - 0 \cdot 2, 0 \cdot 0 - 3 \cdot 1, 3 \cdot 2 - 1 \cdot 0) = (1, -3, 6); \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1 + 0 + 12 = 13.$
Volume 13 cubic units.

2.3 Expanding along the first row: $2((-1)(-2) - 4 \cdot 2) - 1(0 \cdot (-2) - 4 \cdot 5) + 3(0 \cdot 2 - (-1) \cdot 5) = 2(-6) + 20 + 15 = 23.$

2.4 $\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0$ gives $\lambda = 3$ or 1 . $\lambda = 3$: $(\mathbf{M} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $v_1 = v_2$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = 1$: $v_1 = -v_2$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Normalising, $\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (or with columns swapped/negated), and $\mathbf{P}^T \mathbf{A} \mathbf{P} = \text{diag}(3, 1).$

3.1 (a) $(7, 5, -6) \times (-3, 1, 2) = (5 \cdot 2 - (-6)(1), (-6)(-3) - 7 \cdot 2, 7 \cdot 1 - 5 \cdot (-3)) = (16, 4, 22) = 2(8, 2, 11).$ (b) The normal is $(8, 2, 11)$; substituting the anchor point: $p = 4 \cdot 8 + 2 \cdot 2 + (-1) \cdot 11 = 32 + 4 - 11 = 25.$ (c) l_2 has direction $\mathbf{b}_2 = (3, 0, 2) - (2, 1, 3) = (1, -1, -1).$ Common normal: $(13, 3, -10) \times (1, -1, -1) = (3 \cdot (-1) - (-10)(-1), (-10)(1) - 13 \cdot (-1), 13 \cdot (-1) - 3 \cdot 1) = (-13, 3, -16),$ with $|(-13, 3, -16)| = \sqrt{434}.$ Offset between anchors: $(2, 1, 3) - (\frac{13}{5}, -\frac{17}{5}, 1) = (-\frac{3}{5}, \frac{22}{5}, 2).$ Distance = $\frac{|\frac{39}{5} + \frac{66}{5} - 32|}{\sqrt{434}} = \frac{11}{\sqrt{434}}.$

3.2 (a) Row 1: $\mathbf{M}\mathbf{v}$'s first component is $1 - 2 + 4 = 3,$ and $3 = \lambda \cdot 1,$ so $\lambda = 3.$ (b) Row 2: $3 + 2a + b = 3 \times 2 = 6;$ row 3: $a + 2 + b = 3 \times 1 = 3.$ Subtracting: $a = 2;$ then $b = -1.$ (c) With $a = 2, b = -1:$ $(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0}$ has rows $(0, -1, 4)$ and $(3, 1, -1)$ (and $(2, 1, -2)).$ From row 1: $v_2 = 4v_3;$ row 2: $3v_1 + v_2 - v_3 = 0,$ so with $v_2 = 4v_3, 3v_1 = -3v_3$ and $v_1 = -v_3.$ Taking $v_3 = 1:$ eigenvector $(-1, 4, 1).$

Local review guidance: cross products with the correct cyclic signs (middle term negated); plane constant from an anchor point; eigen checks compare component-wise with $\lambda v_j;$ distances projected onto the common normal with exact surds. This is authored classroom guidance.