6.3 Euler's relation, de Moivre's theorem and complex roots

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS Euler's relation • de Moivre's theorem • modulus • argument

Euler's relation, de Moivre's theorem and complex roots

Objective

Use Euler's relation and de Moivre's theorem to expand multiple-angle identities and powers of trigonometric functions, prove the theorem for integer n, and find all nth roots of a complex number in exact polar form (FP2 3.1–3.2).

1 Key knowledge and worked examples

Core relations. $e^{i\theta} = \cos\theta + i\sin\theta$, so $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$. De Moivre: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, provable for positive integers n by induction (the inductive step multiplies by $\cos\theta + i\sin\theta$ and expands with the addition formulae); negative integers follow by taking reciprocals.

Worked example – powers to multiple angles (WFM02 June 2025 Q9). The imaginary part of $(\cos\theta + i\sin\theta)^5$ is $5c^4s - 10c^2s^3 + s^5$; substituting $c^2 = 1 - s^2$ gives $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. Setting $5\theta = \pi$ yields $16s^4 - 20s^2 + 5 = 0$, so $s^2 = \frac{5\pm\sqrt{5}}{8}$: $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$ (the smaller root, since $\sin\frac{\pi}{5} < \sin\frac{3\pi}{5}$).

Worked example – fifth roots (WFM02 June 2025 Q1). For $z^5 = -2\sqrt{2} - 2\sqrt{6}i$: modulus $\sqrt{8+24} = 4\sqrt{2}$, argument $-\frac{2\pi}{3}$ (third quadrant, tangent $\sqrt{3}$). So $z = \sqrt{2}\,e^{i\left(-\frac{2\pi}{15} + \frac{2\pi k}{5}\right)}$, $k = 0, 1, 2, 3, 4$ – five roots spaced $\frac{2\pi}{5}$ apart on $|z| = \sqrt{2}$. Exact polar form only.

2 Practice

2.1 Use de Moivre's theorem to express $\cos 3\theta$ as a polynomial in $\cos\theta$. [3]

2.2 Express $\sin^3\theta$ in the form $a\sin\theta + b\sin 3\theta$. [3]

2.3 Solve $z^3 = 8i$, giving the roots in the form $re^{i\theta}$ with exact θ. [4]

3 Exam-style questions

3.1 *(Original.)* (a) Show that $\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (b) Hence evaluate $\displaystyle\int_0^{\pi/2} \cos^4\theta\, d\theta$. [7]

3.2 *(Edexcel IAL WFM02 June 2025 Q1.)* (a) Express $-2\sqrt{2}-2\sqrt{6}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. (b) Hence solve $z^5 = -2\sqrt{2} - 2\sqrt{6}i$, giving your answers in the form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$. [8]