Solutions

2.1 $\mathbf{AB} = \begin{pmatrix} 1 & -3 \\ 4 & -2 \end{pmatrix}$; $\det \mathbf{A} = 10$; $\mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$.

2.2 (a) Reflection in $y = x$; (b) enlargement, scale factor 3, centre O; (c) stretch, factor 2, parallel to the x-axis; (d) rotation 30° anticlockwise about O.

2.3 Image area $= 2 \times 7 = 14$; the negative determinant means the transformation reverses orientation (includes a reflection).

2.4 $\sum(r^2+2r+1) = \frac{n(n+1)(2n+1)}{6} + n(n+1) + n = \frac{n(n+1)(2n+1)+6n(n+1)+6n}{6} = \frac{n(2n^2+3n+1+6n+6+6)}{6} = \frac{n(2n^2+9n+13)}{6}$ ✓ (check $n = 1$: 4 and $\frac{24}{6} = 4$).

2.5 $n = 1$: both sides $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ✓. If true for k: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$ ✓. True for $n = 1$ and $n = k \Rightarrow n = k+1$; by induction, true for all positive integers.

3.1 (a) $\det \mathbf{M} = -3 - 2a$, so $\mathbf{M}^{-1} = \dfrac{1}{-(3+2a)}\begin{pmatrix} -3 & -a \\ -2 & 1 \end{pmatrix}$. (b) $2\mathbf{M} + 4\mathbf{I} = \begin{pmatrix} 6 & 2a \\ 4 & -2 \end{pmatrix}$. Entry $(2,2)$: $\dfrac{1}{-(3+2a)} = -2 \Rightarrow 3 + 2a = \frac{1}{2} \Rightarrow a = -\frac{5}{4}$. Every other entry confirms it: $(2,1)$: $\dfrac{-2}{-1/2} = 4$ ✓; $(1,1)$: $\dfrac{-3}{-1/2} = 6$ ✓; $(1,2)$: $\dfrac{-a}{-1/2} = 2a$ ✓ (identically true).

3.2 (a) $2\sum_1^{2n} r^2 - 2n = \dfrac{2(2n)(2n+1)(4n+1)}{6} - 2n = \dfrac{2n(2n+1)(4n+1) - 6n}{3} = \dfrac{2n(8n^2+6n-2)}{3} = \dfrac{4n(4n^2+3n-1)}{3} = \frac{4}{3}n(n+1)(4n-1)$ (since $4n^2 + 3n - 1 = (n+1)(4n-1)$). (b) Base $n = 1$: $1 = \frac{1}{4} \cdot 1 \cdot 2 \cdot 2$ ✓. Step: adding $(k+1)(3k+1)^2$ to $\frac{1}{4}k^2(k+1)(9k-7)$ gives $\frac{1}{4}(k+1)[9k^3+29k^2+24k+4] = \frac{1}{4}(k+1)(k+1)(k+2)(9k+2)$, the target for $n = k+1$; close with the induction statement. (c) $8 \cdot \frac{1}{4}n^2(n+1)(9n-7) = 15 \cdot \frac{4}{3}n(n+1)(4n-1)$ – cancel $n(n+1)$ (positive): $2n(9n-7) = 20(4n-1) \Rightarrow 9n^2 - 47n + 10 = 0 \Rightarrow (9n-2)(n-5) = 0$, so $n = 5$ (n a positive integer).

Local review guidance: inverse formula and entry-by-entry comparison; exact values; induction with base case, inductive step and closing sentence. This is authored classroom guidance.

3.3

(a)

- $(3,0) \mapsto (0,3)$ and $(1,2) \mapsto (-2,1)$: a **rotation of** 90° **anticlockwise about** O
- the columns of $\mathbf{P}$ are the images of $\binom{1}{0}$ and $\binom{0}{1}$, namely $\binom{0}{1}$ and $\binom{-1}{0}$: $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b)

- $\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\mathbf{QP}$ means apply $\mathbf{P}$ first, then $\mathbf{Q}$:

$$\mathbf{QP} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- a **reflection in the** x**-axis**

(c)

- area of $T = \frac{1}{2} \times 2 \times 2 = 2$; the area scale factor is $|\det(3\mathbf{QP})|$:

$$\det(3\mathbf{QP}) = 3^2 \det(\mathbf{QP}) = 9(-1) = -9 \;\Rightarrow\; \text{area} = 9 \times 2 = \mathbf{18}$$