3.6 Integration and numerical methods

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS recognition forms • sign change • iteration

Objective

Integrate e^{kx}, $\frac{1}{x}$ and $\sin / \cos kx$; recognise $\int \frac{f'(x)}{f(x)}dx$ and $\int f'(x)(f(x))^n dx$; use identities on squared trig functions; locate roots by sign change; and run iterations $x_{n+1} = f(x_n)$. Original teaching questions for specification items P3.5.1, P3.5.2, P3.6.1 and P3.6.2; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

$$\int e^{kx}dx = \tfrac{1}{k}e^{kx} + c \qquad \int \frac{1}{x}dx = \ln|x| + c$$

$$\int \cos kx\, dx = \tfrac{1}{k}\sin kx + c \qquad \int \sin kx\, dx = -\tfrac{1}{k}\cos kx + c$$

Recognition forms :

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n\,dx = \frac{(f(x))^{n+1}}{n+1} + c \quad (n \neq -1)$$

For squares: $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$, $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$, $\tan^2 A = \sec^2 A - 1$.

Numerical methods : if f is continuous and $f(a)f(b) < 0$, a root lies in (a, b); an iteration $x_{n+1} = f(x_n)$ from a start value approaches a fixed point when it converges – confirm the final accuracy with a sign change.

Worked example – identity first (adapted from Edexcel IAL WMA13 June 2025 Q5). Find $\int \sin^2 3x\, dx$.

- $\cos 6x \equiv 1 - 2\sin^2 3x$, so $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$:

$$\int \sin^2 3x\, dx = \tfrac{1}{2}\int (1 - \cos 6x)\, dx = \frac{x}{2} - \frac{\sin 6x}{12} + c$$

- Check: differentiating gives $\frac{1}{2} - \frac{1}{2}\cos 6x = \sin^2 3x$ ✓.

Worked example – recognition. Find $\displaystyle\int \frac{4x}{x^2+3}\,dx$ and $\displaystyle\int 2x\sqrt{x^2+3}\,dx$.

- First: numerator $4x$ is the derivative of x^2+3:

$$\int \frac{4x}{x^2+3}\,dx = \ln|x^2+3| + c$$

- Second: with $u = x^2+3$, $du = 2x\,dx$:

$$\int 2x\,u^{1/2}\,dx = \int u^{1/2}\,du = \tfrac{2}{3}u^{3/2} + c = \tfrac{2}{3}(x^2+3)^{3/2} + c$$

- Check the second: $\dfrac{d}{dx}\,\frac{2}{3}(x^2+3)^{3/2} = \frac{2}{3}\cdot\frac{3}{2}(x^2+3)^{1/2}\cdot 2x = 2x\sqrt{x^2+3}$ ✓.

2 Practice

2.1 Find $\displaystyle\int e^{5x}\,dx$. [2]

2.2 Find $\displaystyle\int \frac{3}{x}\,dx$ $(x > 0)$. [2]

2.3 Find $\displaystyle\int \cos 4x\,dx$. [2]

2.4 Find $\displaystyle\int \frac{2x+2}{x^2+2x+5}\,dx$. [3]

2.5 Find $\int 3x^2(x^3+1)^4\,dx$. [3]

2.6 Show that $x^3 - 3x + 1 = 0$ has a root between 0 and 1. [2]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA13 June 2025 Q5 and Q7.)*

(a) Find $\int \cos^2 2x\,dx$.

(b) Find $\int \frac{x}{x^2+4}\,dx$. [5]

3.2 *(Adapted from Edexcel IAL WMA13 June 2025 Q7.)* The equation $x = \mathrm{e}^{-x}$ has exactly one positive root.

(a) Show the root lies between 0.5 and 0.6.

(b) Use the iteration $x_{n+1} = \mathrm{e}^{-x_n}$ from $x_1 = 0.5$ to find x_3 to 3 decimal places.

(c) Verify the root is 0.567 to 3 decimal places. [7]