Solutions

2.1 $\dfrac{dy}{dx} = 5\cos 5x$.

2.2 $\dfrac{dy}{dx} = -3\mathrm{e}^{-3x}$.

2.3 $\dfrac{dy}{dx} = 2x\cos x - x^2\sin x$.

2.4 $\dfrac{dy}{dx} = \dfrac{\mathrm{e}^x \cdot x - \mathrm{e}^x \cdot 1}{x^2} = \dfrac{\mathrm{e}^x(x-1)}{x^2}$.

2.5 $\dfrac{dy}{dx} = \dfrac{12x^2}{4x^3+1}$ (chain: outer ln, inner $4x^3+1$).

2.6 $\dfrac{dx}{dy} = 2\mathrm{e}^{2y} = 2x$, so $\dfrac{dy}{dx} = \dfrac{1}{2x}$.

3.1 From the worked derivative: stationary points have

$$\mathrm{e}^{-x^2}\left(3\cos 3x - 2x\sin 3x\right) = 0$$

$\mathrm{e}^{-x^2} > 0$ always, so $3\cos 3x = 2x\sin 3x$. Dividing by $\cos 3x$ (at a stationary point $\cos 3x \neq 0$, since that would force $\sin 3x = \pm 1 \neq 0$ and then $0 = \pm 2x$ fails for $x > 0$):

$$3 = 2x\tan 3x \implies \tan 3x = \frac{3}{2x} \implies 3x = \arctan\frac{3}{2x} \implies x = \tfrac{1}{3}\arctan\frac{3}{2x} \qquad \blacksquare$$

3.2 $\dfrac{dV}{dt} = -50\mathrm{e}^{-0.1t}$; at $t = 10$: $-50\mathrm{e}^{-1} = -18.4$ litres per minute (leaving at 18.4 L/min). Rate $= 5$: $50\mathrm{e}^{-0.1t} = 5 \Rightarrow \mathrm{e}^{-0.1t} = 0.1 \Rightarrow t = 10\ln 10 = 23.0$ minutes (3 s.f.). Check: $V(23.0) = 500\mathrm{e}^{-2.303} = 50.0$ litres, one tenth of the initial – matching the rate being one tenth of the initial 50 L/min ✓.

Local review guidance: correct rule with both parts differentiated; simplified derivative; exponential-rate equation solved with logs. This is authored classroom guidance.

3.3

(a)

- product rule, $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$, with $u = 2x$, $v = \ln x$:

$$\frac{dy}{dx} = 2x \cdot \frac{1}{x} + 2\ln x = 2\ln x + 2$$

(b)

- at $x = \mathrm{e}$: $y = 2\mathrm{e}\ln\mathrm{e} = 2\mathrm{e}$, gradient $m = 2\ln\mathrm{e} + 2 = 4$
- $y - y_1 = m(x - x_1)$: $y - 2\mathrm{e} = 4(x - \mathrm{e})$, so $\boldsymbol{y = 4x - 2\mathrm{e}}$

(c)

- at $x = 0$ the tangent gives $A(0,\ -2\mathrm{e})$
- the normal has gradient $-\frac{1}{4}$: $y - 2\mathrm{e} = -\frac{1}{4}(x - \mathrm{e})$; at $x = 0$, $y = 2\mathrm{e} + \frac{1}{4}\mathrm{e} = \frac{9}{4}\mathrm{e}$, so $B(0,\ \frac{9}{4}\mathrm{e})$
- base AB on the y-axis, height the x-coordinate of P, area $= \frac{1}{2} \times AB \times \mathrm{e}$:

$$\text{area} = \tfrac{1}{2}\left(\tfrac{9}{4}\mathrm{e} + 2\mathrm{e}\right)\mathrm{e} = \tfrac{1}{2} \cdot \tfrac{17}{4}\mathrm{e} \cdot \mathrm{e} = \boldsymbol{\frac{17}{8}\mathrm{e}^2}$$