# 2.1 Proof: exhaustion and counterexample

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

**KEY WORDS** proof • proof by exhaustion • counterexample

## Objective

Structure mathematical proofs, use proof by exhaustion over complete sets of cases, and disprove statements with a counterexample. Original teaching questions for specification items P2.1.1 to P2.1.3; the exam-style part adapts reviewed WMA12 June 2025 demand.

## 1 Key knowledge and worked examples

A **proof** starts from given assumptions, moves by logical steps, and ends with the target conclusion. Quote every assumption you use; do not assume what you are proving.

- **Proof by exhaustion** lists every possible case and checks each one. The case list must be complete – "every integer is of the form $3q$, $3q+1$ or $3q+2$" is complete; "the first few integers" is not.
- **Disproof by counterexample** needs exactly one example that breaks the claim.

**Worked example – exhaustion.** Prove that for every integer $n$, the number $n^2+n$ is even.

- The cases split on parity, and every integer is in one case:
- $n$ even, $n=2k$: $n^2+n=4k^2+2k=2(2k^2+k)$ – even.
- $n$ odd, $n=2k+1$: $n^2+n=(2k+1)(2k+2)$, and $2k+2$ is even, so the product is even.
- Both cases give an even number, so $n^2+n$ is always even. ■
- Check: $n=4$: 20 ✓; $n=5$: 30 ✓; $n=-3$: 6 ✓.

**Worked example – counterexample.** Disprove: "$2^n+1$ is prime for every positive integer $n$."

- Try small values: $3, 5, 9, \ldots$ – and $2^3+1=9=3\times 3$ is not prime.
- One counterexample ($n=3$) disproves the claim for all $n$. ■

## 2 Practice

**2.1** Disprove by counterexample: "$n^2 + 1$ is prime for all positive integers $n$." [2]

**2.2** Prove that the sum of three consecutive integers is divisible by 3. [3]

**2.3** Prove by exhaustion that $n^2 - 1$ is divisible by 3 for every positive integer $n$ that is not a multiple of 3. [4]

**2.4** Prove that the product of two odd integers is odd. [3]

## 3 Exam-style questions

**3.1** *(Adapted from Edexcel IAL WMA12 June 2025 Q10.)* Use algebra to prove by exhaustion that for all positive integers $m$ that are not multiples of 3, the value of $m^2 + 3m + 2$ is always a multiple of 3. [4]

**3.2** Prove by exhaustion that no perfect square ends in the digit 8. [4]