

7.6 Vectors and further matrix algebra

Name: _____ Class: _____ Date: _____

Total: 25 marks

KEY WORDS vector product • scalar triple product • eigenvalue
• eigenvector

Vectors and further matrix algebra

Objective

Compute vector and scalar triple products with their geometric meanings, convert between line and plane forms, find point-to-plane and skew-line distances, and handle 3×3 determinants, inverses and eigenvalues (FP3 5.1–5.3, 6.4–6.8).

1 Key knowledge and worked examples

Vectors. $\mathbf{a} \times \mathbf{b}$ via the determinant with $\mathbf{i}, \mathbf{j}, \mathbf{k}$; its modulus is the parallelogram area. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the parallelepiped volume (zero $\iff$ coplanar). Planes: $\mathbf{r} \cdot \mathbf{n} = p$ (normal from $\mathbf{b} \times \mathbf{c}$ in the parametric form). Lines: $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. Shortest distance between skew lines with directions $\mathbf{b}_1, \mathbf{b}_2$ through $\mathbf{a}_1, \mathbf{a}_2$: $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$.

Matrices. det by row/column expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$. Eigenvector check row by row: each component of $\mathbf{M}\mathbf{v}$ equals λ times the matching component of $\mathbf{v}$ – not λ itself. For symmetric matrices, normalised eigenvectors form an orthogonal $\mathbf{P}$ with $\mathbf{P}^T \mathbf{A} \mathbf{P}$ diagonal.

Worked example – eigenvalues from a given eigenvector (WFM03 June 2025 Q3). $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$, eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Row 1: $1 - 2 + 4 = 3 = 3(1)$, so $\lambda = 3$. Row 2: $3 + 2a + b = 3(2) = 6$; row 3: $a + 2 + b = 3(1) = 3$. Solving: $a = 2$, $b = -1$. The trace (2) and determinant (–6) then force the other eigenvalues 1 and –2, with eigenvectors $(-1, 4, 1)$ and $(-1, 1, 1)$.

2 Practice

2.1 Find $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$. [2]

2.2 Find the volume of the parallelepiped spanned by $\mathbf{a} = (1, 0, 2)$, $\mathbf{b} = (3, 1, 0)$ and $\mathbf{c} = (0, 2, 1)$. [3]

2.3 Evaluate the determinant of $\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & -2 \end{pmatrix}$. [3]

2.4 Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and write down an orthogonal matrix $\mathbf{P}$ that diagonalises it. [4]

3 Exam-style questions

3.1 (*Edexcel IAL WFM03 June 2025 Q8, condensed.*) The plane Π_1 has equation $\mathbf{r} = (4, 2, -1) + \lambda(7, 5, -6) + \mu(-3, 1, 2)$. (a) Determine $(7\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) \times (-3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. (b) Hence show that the equation of Π_1 can be written as $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = p$, stating the value of p . Given that the line of intersection of Π_1 with the plane $\Pi_2 : x - y + z = 7$ has direction $(13, 3, -10)$ and passes through $(\frac{13}{5}, -\frac{17}{5}, 1)$, and that the line l_2 passes through $A(2, 1, 3)$ and $B(3, 0, 2)$, (c) determine the shortest distance between the two lines. [7]

3.2 (*Edexcel IAL WFM03 June 2025 Q3, condensed.*) $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$, where a and b are constants, has eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. (a) Determine the corresponding eigenvalue. (b) Hence determine a and b . (c) Given that the other eigenvalues of $\mathbf{M}$ are 1 and -2 , determine an eigenvector corresponding to $\lambda = 1$. [6]

Solutions

2.1 $\mathbf{i}((-3)(-2) - (1)(4)) - \mathbf{j}((2)(-2) - (1)(1)) + \mathbf{k}((2)(4) - (-3)(1)) = 2\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}.$

2.2 $\mathbf{b} \times \mathbf{c} = (1 \cdot 1 - 0 \cdot 2, 0 \cdot 0 - 3 \cdot 1, 3 \cdot 2 - 1 \cdot 0) = (1, -3, 6); \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1 + 0 + 12 = 13.$
Volume 13 cubic units.

2.3 Expanding along the first row: $2((-1)(-2) - 4 \cdot 2) - 1(0 \cdot (-2) - 4 \cdot 5) + 3(0 \cdot 2 - (-1) \cdot 5) = 2(-6) + 20 + 15 = 23.$

2.4 $\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0$ gives $\lambda = 3$ or 1 . $\lambda = 3$: $(\mathbf{M} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $v_1 = v_2$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = 1$: $v_1 = -v_2$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Normalising, $\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (or with columns swapped/negated), and $\mathbf{P}^T \mathbf{A} \mathbf{P} = \text{diag}(3, 1).$

3.1 (a) $(7, 5, -6) \times (-3, 1, 2) = (5 \cdot 2 - (-6)(1), (-6)(-3) - 7 \cdot 2, 7 \cdot 1 - 5 \cdot (-3)) = (16, 4, 22) = 2(8, 2, 11).$ (b) The normal is $(8, 2, 11)$; substituting the anchor point: $p = 4 \cdot 8 + 2 \cdot 2 + (-1) \cdot 11 = 32 + 4 - 11 = 25.$ (c) l_2 has direction $\mathbf{b}_2 = (3, 0, 2) - (2, 1, 3) = (1, -1, -1).$ Common normal: $(13, 3, -10) \times (1, -1, -1) = (3 \cdot (-1) - (-10)(-1), (-10)(1) - 13 \cdot (-1), 13 \cdot (-1) - 3 \cdot 1) = (-13, 3, -16),$ with $|(-13, 3, -16)| = \sqrt{434}.$ Offset between anchors: $(2, 1, 3) - (\frac{13}{5}, -\frac{17}{5}, 1) = (-\frac{3}{5}, \frac{22}{5}, 2).$ Distance = $\frac{|\frac{39}{5} + \frac{66}{5} - 32|}{\sqrt{434}} = \frac{11}{\sqrt{434}}.$

3.2 (a) Row 1: $\mathbf{M}\mathbf{v}$'s first component is $1 - 2 + 4 = 3,$ and $3 = \lambda \cdot 1,$ so $\lambda = 3.$ (b) Row 2: $3 + 2a + b = 3 \times 2 = 6;$ row 3: $a + 2 + b = 3 \times 1 = 3.$ Subtracting: $a = 2;$ then $b = -1.$ (c) With $a = 2, b = -1:$ $(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0}$ has rows $(0, -1, 4)$ and $(3, 1, -1)$ (and $(2, 1, -2)).$ From row 1: $v_2 = 4v_3;$ row 2: $3v_1 + v_2 - v_3 = 0,$ so with $v_2 = 4v_3, 3v_1 = -3v_3$ and $v_1 = -v_3.$ Taking $v_3 = 1:$ eigenvector $(-1, 4, 1).$

Local review guidance: cross products with the correct cyclic signs (middle term negated); plane constant from an anchor point; eigen checks compare component-wise with $\lambda v_j;$ distances projected onto the common normal with exact surds. This is authored classroom guidance.