6.5 First and second order differential equations

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS integrating factor • complementary function • particular integral • auxiliary equation

First and second order differential equations

Objective

Solve first order linear equations with an integrating factor, carry out given substitutions that reduce an equation to linear or separable form, and solve second order linear equations with constant coefficients via the auxiliary equation and a trial particular integral, including the resonance case, then fit initial conditions (FP2 4.1–4.3, 5.1–5.2).

1 Key knowledge and worked examples

Integrating factor. For $\dfrac{dy}{dx} + Py = Q$: multiply by $e^{\int P\,dx}$ so the left side becomes $\dfrac{d}{dx}\left(ye^{\int P\,dx}\right)$, then integrate once. For a given substitution $z = f(y)$, compute $\dfrac{dz}{dx}$ by the chain rule, rewrite, solve, substitute back – and answer in the form the question names. If y^2 is determined, y has two signs.

Worked example – substitution then integrating factor (WFM02 June 2025 Q6). $2\dfrac{dy}{dx} + y\cot x + y^3 \tan x \sec x = 0$; with $z = \frac{1}{y^2}$, $2\dfrac{dy}{dx} = -y^3\dfrac{dz}{dx}$, and dividing by y^3 gives $\dfrac{dz}{dx} - z\cot x = \tan x \sec x$. IF $= e^{-\int \cot x\,dx} = \csc x$; $\dfrac{d}{dx}(z\csc x) = \sec^2 x$, so $z = \sin x \tan x + C\sin x$ and $y^2 = \dfrac{1}{\sin x \tan x + C\sin x}$. Given $y^2 = \frac{4\sqrt{3}}{3}$ at $x = \frac{\pi}{6}$: $C = \frac{1}{2\sqrt{3}}$, and at $x = \frac{\pi}{3}$, $z = \frac{7}{4}$, so $y = \pm\dfrac{2}{\sqrt{7}}$.

Worked example – second order with resonance (WFM02 June 2025 Q4). Given that $y = \lambda x e^{3x}$ is a particular integral of $4y'' - 11y' - 3y = 78e^{3x}$: substituting gives $13\lambda e^{3x} = 78e^{3x}$, so $\lambda = 6$. Auxiliary $4m^2 - 11m - 3 = (4m+1)(m-3) = 0$: $y = Ae^{-x/4} + Be^{3x} + 6xe^{3x}$. With $y = \frac{9}{2}$, $y' = 0$ at $x = 0$: $A + B = \frac{9}{2}$, $-\frac{A}{4} + 3B + 6 = 0$,

so $A = 6$, $B = -\frac{3}{2}$ and $y = 6e^{-x/4} - \frac{3}{2}e^{3x} + 6xe^{3x}$.

2 Practice

2.1 Solve $\dfrac{dy}{dx} + 2y = e^{-x}$. [4]

2.2 Solve $x\dfrac{dy}{dx} + y = x^2$ for $x > 0$. [3]

2.3 Solve $y'' + y = 0$ given $y = 1$ and $y' = 0$ at $x = 0$. [2]

2.4 Find the general solution of $y'' - 4y = 12x$. [4]

3 Exam-style questions

3.1 *(Original.)* Find the general solution of $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-2x}$. [6]

3.2 *(Edexcel IAL WFM02 June 2025 Q4, condensed.)* Given that $y = \lambda x e^{3x}$ is a particular integral of the differential equation $4\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 3y = 78e^{3x}$, (a) determine the value of λ. (b) Hence determine the general solution of the differential equation. Given also that $y = \frac{9}{2}$ and $\dfrac{dy}{dx} = 0$ at $x = 0$, (c) determine the particular solution. [6]

Solutions

2.1 IF $= e^{2x}$: $\dfrac{d}{dx}(ye^{2x}) = e^{2x}e^{-x} = e^{x}$, so $ye^{2x} = e^{x} + C$ and $y = e^{-x} + Ce^{-2x}$.

2.2 The left side is $\dfrac{d}{dx}(xy)$: $xy = \frac{x^3}{3} + C$, so $y = \frac{x^2}{3} + \dfrac{C}{x}$.

2.3 Auxiliary $m^2 + 1 = 0$: $y = A\cos x + B\sin x$. $y(0) = A = 1$; $y' = -A\sin x + B\cos x$, $y'(0) = B = 0$. So $y = \cos x$.

2.4 Auxiliary $m^2 - 4 = 0$: $m = \pm 2$, CF $= Ae^{2x} + Be^{-2x}$. Trial PI $= ax + b$: substitution gives $-4(ax + b) = 12x$, so $a = -3$, $b = 0$. General solution: $y = Ae^{2x} + Be^{-2x} - 3x$.

3.1 Auxiliary $m^2 + 4m + 4 = (m + 2)^2 = 0$: repeated root -2, so CF $= (A + Bx)e^{-2x}$. Since e^{-2x} and xe^{-2x} solve the homogeneous equation, try PI $= \frac{\lambda x^2}{2}e^{-2x}$: with $y = \frac{\lambda x^2}{2}e^{-2x}$, $y' = \lambda(x - x^2)e^{-2x}$ and $y'' = \lambda(2x^2 - 4x + 1)e^{-2x}$. Combining: $y'' + 4y' + 4y = \lambda e^{-2x}\left[(2x^2 - 4x + 1) + 4(x - x^2) + 2x^2\right] = \lambda e^{-2x}$. So $\lambda = 1$ and the general solution is $y = (A + Bx)e^{-2x} + \frac{x^2}{2}e^{-2x}$.

3.2 (a) $y' = \lambda(1 + 3x)e^{3x}$, $y'' = \lambda(6 + 9x)e^{3x}$, so $4y'' - 11y' - 3y = \lambda e^{3x}(24 + 36x - 11 - 33x - 3x) = 13\lambda e^{3x} = 78e^{3x}$, hence $\lambda = 6$. (b) $4m^2 - 11m - 3 = (4m + 1)(m - 3)$, so $m = -\frac{1}{4}$ or 3: $y = Ae^{-x/4} + Be^{3x} + 6xe^{3x}$. (c) $y(0) = A + B = \frac{9}{2}$; $y' = -\frac{A}{4}e^{-x/4} + 3Be^{3x} + 6(1 + 3x)e^{3x}$, so $y'(0) = -\frac{A}{4} + 3B + 6 = 0$. From the first, $A = \frac{9}{2} - B$; substituting: $-\frac{1}{4}(\frac{9}{2} - B) + 3B = -6$, so $-\frac{9}{8} + \frac{13B}{4} = -6$, $\frac{13B}{4} = -\frac{39}{8}$, $B = -\frac{3}{2}$, and $A = 6$. Particular solution: $y = 6e^{-x/4} - \frac{3}{2}e^{3x} + 6xe^{3x}$.

Local review guidance: correct IF or auxiliary treatment; trial PI adjusted for resonance when the plain trial dies; constants found from the full general solution, not the CF. This is authored classroom guidance.